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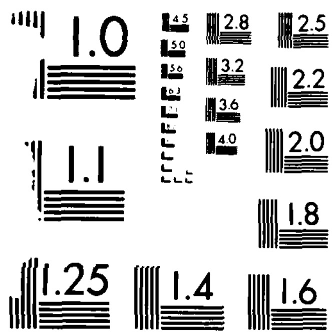

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A STUDY OF THE OPTIMIZATION PROBLEM FOR CALIBRATING A LACOSTE AND ROMBERG "G" GRAVITY METER TO DETERMINE CIRCULAR ERRORS

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This report discusses how the circular errors of a gravity meter could be effectively calibrated in a laboratory. An optimization method is used in this study. Minimization of the trace of the variance-covariance matrix of adjusted parameters, Σ_x, is adopted as the criterion for the optimization. The mathematical analysis of the trace is made in the case of one wavelength in order to find the best distribution of observations, as well as

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(20) the worst. For several wavelengths, a number of simulative computations are carried out by solving the equations $\partial TR(\Sigma_x)/\partial\Delta d = 0$, $\partial TR(\Sigma_x)/\partial d_{off} = 0$ and $\partial TR(\Sigma_x)/\partial W = 0$ (d_{off} = dial reading weight off, $\Delta d = (d_{off})_{i+1} - (d_{off})_i$, W = calibrating weight) for finding effective distribution of observations and the best weights, as well as the worst. A set of numerical solutions for the equations over a certain range of observations is obtained.

Based on the simulative studies, the concepts of phase distribution and effectiveness of observations in the periodic error calibration are presented and so a design for the most effective distribution of observations is introduced. For the calibration of periodic errors with several wavelengths, it is preferable to select two weights that can be mutually compensated in fitting them with all involved periods. Some possible compensative weights for an LCR "G" gravity meter with periodic errors of 1206, 1206/17, 1206/34, and 1 counter units (c.u.) are presented.

An attempt is made to answer how many observations should be made for determining the periodic screw errors with reasonable accuracy.

FORWARD

This report was prepared in part by Dingbo Chao while on leave from the Wuhan Technical University of Surveying and Mapping, Wuhan, The People's Republic of China, and in part by Edward M. Baker, a graduate research associate working under Air Force Contract No. F19628-82-K-0017, The Ohio State University Research Foundation, Project No. 714255, Project Supervisor, Urho A. Uotila, Professor, Department of Geodetic Science and Surveying. The contract covering this research is administered by The Air Force Geophysics Laboratory (AFGL), Hanscom Air Force Base, Massachusetts, with Christopher Jekeli, Contract Manager.

TABLE OF CONTENTS

PAGE

LIST OF TABLES

LIST OF FIGURES

1. INTRODUCTION

It has been verified by studies and experiments within the last 10 years that the periodic screw error is one of the major instrumental error sources of the Lacoste-Romberg (LCR) model "G" and model "D" gravity meters (Torge and Kanngieser, 1979, Kanngieser and Torge, 1981, Lambert and Liard, 1981, Becker, 1981). The magnitudes of the errors range from a few μgal for short wavelengths to hundreds of μgal for long wavelengths (Kanngieser and Torge, 1981). It has become important to determine these periodic screw errors in laboratory calibration, and then to eliminate or weaken the errors while processing the data.

An apparatus called "Cloudcroft, Jr." is used in laboratory calibration (Krieg, 1981, p. 29). The apparatus includes two weights. The larger of the two is referred to as the 180 mgal weight and the smaller one is the 20 mgal weight. Using this apparatus, changes of approximately 20, 180, and 200 counter units can be induced in the gravity meter by placing the weights in so-called "weight-on" and "weight-off" positions. In one operation for weight-off and weight-on, a pair of observations is made at one location within the range of counter readings. Subsequent observation pairs are made at different locations. Usually, the positions of the observations are distributed with nearly equal spacing over a range of counter readings.

Because of the nature of periodic functions, the accuracy of the determined parameters of a periodic screw error depends not only on how many observations are made, but also on what size of calibrating weight is used and the distribution of observations. If the observations were made with a certain special distribution or a special size of weight (for instance, with a distribution of equally-spaced intervals or a weight equal to an integral

multiple of the wavelength of one periodic screw error), the errors of the adjusted parameters would tend toward infinity. The accuracy of the calibration might be significantly improved if a proper distribution is arranged and an appropriate size of weight is adopted. Some examples will be given in this report.

Determining how the distribution of observations should be designed and how the size of calibrating weights should be selected to obtain a higher accuracy, while also being economical of expense and time, becomes a problem of optimization. For one wavelength of the periodic screw error, as a special example, a detailed theoretical analysis of the problem is presented in this report, and it provides a basic idea for solving the problem. With respect to the optimization of the calibration of the periodic screw error involving several wavelengths, simulation studies should give ideas of the most effective distribution of observations for laboratory calibration (Uotila, 1982). A practical periodic screw error function for a LCR model "G" gravity meter is given in the simulation study. Also in this report, another ficticious periodic error function is used for simulating and comparing.

2. MATHEMATICAL MODELS

2.1 Mathematical Models for Adjustment

Dial readings are taken as the observables. The function converting dial readings to milligal equivalents and the condition equation with unknown parameters are respectively given by (Uotila, 1982):

$$(1) \qquad D_j = x \cdot d_j + \sum_{i=1}^{k} A_i \cos\left(\frac{2\pi}{T_i} d_j - \phi_i\right) \quad (j = 1, 2, \ldots r)$$

$$(2) \qquad x(d_{on_j} - d_{off_j}) + \sum_{i=1}^{k} A_i\left[\cos\left(\frac{2\pi}{T_i} d_{on_j} - \phi_i\right) - \cos\left(\frac{2\pi}{T_i} d_{off_j} - \phi_i\right)\right] - W = 0$$

where x = linear scale factor (unknown parameter)

T_i = period of screw error

A_i = amplitude (unknown parameter)

ϕ_i = phase lag (unknown parameter)

d_j = dial reading (d_{on_j}: weight on, d_{off_j}: weight off)

D_j = milligal equivalent to d_j

W = calibrating weight

Some other effects, such as the earth tidal effect and drift of instrument are omitted in eq. (2). We assume that their effects on observations are eliminated in advance. Applying appropriate trigonometric identities (2) becomes

$$(3) \qquad (d_{on_j} - d_{off_j}) \cdot x + \sum_{i=1}^{k} \left(\cos \frac{2\pi}{T_i} d_{on_j} - \cos \frac{2\pi}{T_i} d_{off_j}\right) Y_i + \sum_{i=1}^{k} \left(\sin \frac{2\pi}{T_i} d_{on_j} - \sin \frac{2\pi}{T_i} d_{off_j}\right) Z_i - W = 0$$

where $Y_i = A_i \cos\phi_i$

$Z_i = A_i \sin\phi_i$

are two indirect parameters. If A and ϕ are directly taken as parameters, the linearized equation corresponding to (2) is:

$$(4) \qquad (d_{on_j} - d_{off_j})\, x + \sum_{i=1}^{k} \left[\cos\left(\frac{2\pi}{T_i} d_{on_j} - \phi_i\right) - \cos\left(\frac{2\pi}{T_i} d_{off_j} - \phi_i\right)\right]\delta A_i$$

$$+ \sum_{i=1}^{k} \left[\sin\left(\frac{2\pi}{T_i} d_{on_j} - \phi_i\right) - \sin\left(\frac{2\pi}{T_i} d_{off_j} - \phi_i\right)\right] A_i \delta\phi_i + W = 0$$

where W is a constant term. The general form of the condition equations with parameters is as follows:

$$F(L_a, X_a) = 0$$

After linearization, the conditions have the general form:

$$BV + AX + W_o = 0$$

$$\text{where} \quad B = \frac{\partial F}{\partial L_a}, \; A = \frac{\partial F}{\partial X_a}, \; W_o = F(L_b, X_o)$$

The notations used are those of (Uotila, 1967). From (4) we get

$${}_rB_{2r} = X_o \left(\begin{array}{cccc|cccc} 1 & & & 0 & -1 & & & 0 \\ & 1 & & & & -1 & & \\ & & \ddots & & & & \ddots & \\ 0 & & & 1 & 0 & & & -1 \end{array}\right)$$

$${}_rA_{2k+1} = \begin{pmatrix} \Delta d_{01}, & \Delta c_{11}, & \Delta c_{12}, & \cdot & \Delta c_{1k}, & A_1\Delta S_{11}, & A_2\Delta S_{12}, & \cdot & A_k\Delta S_{1k} \\ \Delta d_{02}, & \Delta c_{21}, & \Delta c_{22}, & \cdot & \Delta c_{2k}, & A_1\Delta S_{21}, & A_2\Delta S_{22}, & \cdot & A_k\Delta S_{2k} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \Delta d_{0r}, & \Delta c_{r1}, & \Delta c_{r2}, & \cdot & \Delta c_{rk}, & A_1\Delta S_{r1}, & A_2\Delta S_{r2}, & \cdot & A_k\Delta S_{rk} \end{pmatrix}$$

where

$$\Delta d_{oj} = d_{on_j} - d_{off_j}$$

$$(5) \qquad \Delta C_{ji} = \cos\left(\frac{2\pi}{T_i} d_{on_j} - \phi_i\right) - \cos\left(\frac{2\pi}{T_i} d_{off_j} - \phi_i\right)$$

$$\Delta S_{ji} = \sin\left(\frac{2\pi}{T_i} d_{on_j} - \phi_i\right) - \sin\left(\frac{2\pi}{T_i} d_{off_j} - \phi_i\right)$$

$$(i = 1, 2, \ldots k; \; j = 1, 2, \ldots r)$$

The minimum variance solution of the system of linearized condition equations is obtained from:

$$\hat{X} = -\left(A^T(BB^T)^{-1}A\right)^{-1}A^T(BB^T)^{-1}W_o$$

$$\hat{V} = -B^T(BB^T)^{-1}(AX + W_o)$$

$$\hat{V}^T\hat{V} = -(AX + W_o)^T((BB^T)^{-1})^T W_o$$

$$\Sigma_{\hat{X}} = \sigma_o^2 \, (A^T(BB^T)^{-1}A)^{-1}$$

$$\hat{\sigma}_o^2 = \frac{\hat{V}^T\hat{V}}{DF}$$

$DF = r - 2k - N_w$

r = number of condition equations

N_w = number of calibrating weights

Concerning the above expressions, refer to (Uotila, 1967).

Since all measurements are of the same type, the weight matrix P is assumed to be the unit matrix. If $x_o = 1$, then $BB^T = 2 \cdot {}_rI_r$ and $(BB^T)^{-1} = \frac{1}{2}\,{}_rI_r$. Therefore we get

$$\hat{X} = -(A^TA)^{-1}A^T W_o \tag{6}$$

$$\hat{V} = -\frac{1}{2}B^T(A\hat{X} + W_o) \tag{7}$$

$$\hat{V}^T\hat{V} = \frac{1}{2}\,(A\hat{X} + W_o)^T W_o \tag{8}$$

$$\Sigma_{\hat{X}} = 2\sigma_o^2 \, (A^TA)^{-1} \tag{9}$$

$$TR(\Sigma_{\Delta\hat{a}}) = TR(A\Sigma_{\hat{X}}A^T) = 2N_u\sigma_o^2, \tag{10}$$

where N_u = number of unknowns. Equation (10) can be used for checking the computation. If $N_w > 1$, the unknowns δW should be added. Sometimes δW are neglected because their effects on the accuracy of δA and $\delta\phi$ would be small.

This has been tested by simulative computation. We are interested only in the accuracy of δA and $\delta\phi$.

2.2 Mathematical Models for Optimization

Several criteria could be selected for optimization. It is convenient for mathematical analysis to minimize the trace of Σ_x, that is:

$$(11) \qquad TR(\Sigma_x) = \text{minimum}$$

Hence, the objective function for optimization is as follows:

$f = TR(\Sigma_x) = f(L)$

$L^T = (W, d_{off_1}, d_{off_2}, \ldots d_{off_r})$

or, $L^T = (W, d_{off}, \Delta d_1, \Delta d_2, \ldots \Delta d_{r-1})$

where $\Delta d_j = d_{off_{j+1}} - d_{off_j}$.

Note: each d_{off_j} is considered to be an independent variable. Each d_{on_j}, which is correlated with d_{off_j} and W, is a dependent variable.

The best distribution of observations and the best calibrating weights are those which minimize the objective function. They can be determined by solving the following equations:

$$(12) \qquad \frac{\partial f}{\partial W_n} = 0 \qquad (n = 1, 2, \ldots)$$

$$(13) \qquad \frac{\partial f}{\partial \Delta d_j} = 0 \qquad (j = 1, 2, \ldots r - 1)$$

$$(14) \qquad \frac{\partial f}{\partial d_{off_j}} = 0 \qquad (j = 1, 2, \ldots r)$$

In general, $\frac{\partial f}{\partial L} = \frac{\partial TR(\Sigma_x)}{\partial L} = TR(\frac{\partial \Sigma_x}{\partial L})$. Accordingly, (12), (13), and (14) are expressed by:

(15) $$TR(\frac{\partial \Sigma_x}{\partial L}) = 0$$

From (9), $\frac{\partial \Sigma_x}{\partial L} = \frac{\partial (2\sigma_o{}^2 (A^T A)^{-1})}{\partial L}$

Let $N = A^T A$, then

(16) $$\frac{\partial \Sigma_x}{\partial L} = 2\sigma_o{}^2 \frac{\partial (N^{-1})}{\partial L}$$

Based on differential methods for matrices, we get

(17) $$\frac{\partial (N^{-1})}{\partial L} = N^{-1} \frac{\partial N}{\partial L} N^{-1}$$

(18) $$\frac{\partial N}{\partial L} = \frac{\partial (A^T A)}{\partial L} = \frac{\partial A^T}{\partial L} A + A^T \frac{\partial A}{\partial L}$$

Then, (15) can be expressed by:

(19) $$TR((A^T A)^{-1} (\frac{\partial A^T}{\partial L} A + A^T \frac{\partial A}{\partial L}) (A^T A)^{-1}) = 0$$

It is easy to find the derivatives $\frac{\partial A}{\partial L}$. There are three types of elements in matrix A (See (5)), Δd_{o_j}, ΔC_{ji}, and ΔS_{ji}. Their derivatives with respect to the variables of the objective function W_n, Δd_j and d_{off_j} are as follows:

$$\frac{\partial \Delta d_{o_j}}{\partial d_{off_j}} = \frac{\partial (d_{on_j} - d_{off_j})}{\partial d_{off_j}} = \frac{x - \sum_{i=1}^{k} \lambda_i A_i \sin(\lambda_i d_{off_j} - \phi_i)}{x - \sum_{i=1}^{k} \lambda_i A_i \sin(\lambda_i d_{on_j} - \phi_i)} - 1 = \frac{M_{1j}}{M_{2j}} - 1$$

$$\frac{\partial \Delta d_{o_j}}{\partial \Delta d_j} = \left(\frac{M_{1j}}{M_{2j}} - 1\right) (j - 1) \quad (\Delta d_1 = \Delta d_2 = \dots = \Delta d_r)$$

$$\frac{\partial \Delta d_{o_j}}{\partial W_n} = \frac{1}{M_{2j}}$$

$$\frac{\partial \Delta C_{ij}}{\partial d_{off_j}} = -\lambda_i \sin(\lambda_i d_{on_j} - \phi_i) \frac{M_{1j}}{M_{2j}} + \lambda_i \sin(\lambda_i d_{off_j} - \phi_i)$$

(20) $$\frac{\partial \Delta C_{ij}}{\partial \Delta d_j} = \frac{\partial \Delta C_{ij}}{\partial d_{off_j}} (j - 1) \quad (\Delta d_1 = \Delta d_2 = \dots = \Delta d_r)$$

$$\frac{\partial \Delta C_{ij}}{\partial W_n} = - \frac{\lambda_i \sin(\lambda_i d_{on_j} - \phi_i)}{M_{2j}}$$

$$\frac{\partial \Delta S_{ij}}{\partial d_{off_j}} = \lambda_i [\cos(\lambda_i d_{on_j} - \phi_i) \frac{M_{1j}}{M_{2j}} - \cos(\lambda_i d_{off_j} - \phi_i)]$$

$$\frac{\partial \Delta S_{ij}}{\partial \Delta d_j} = \frac{\partial \Delta S_{ij}}{\partial d_{off_j}} (j - 1) \quad (\Delta d_1 = \Delta d_2 = \dots = \Delta d_r)$$

$$\frac{\partial \Delta S_{ij}}{\partial W_n} = \frac{\lambda_i \cos(\lambda_i d_{on_j} - \phi_i)}{M_{2j}}$$

where $\lambda_i = \frac{2\pi}{T_i}$

$$M_{1j} = x - \sum_{i=1}^{k} \lambda_i A_i \sin(\lambda_i d_{off_j} - \phi_i)$$

$$M_{2j} = x - \sum_{i=1}^{k} \lambda_i A_i \sin(\lambda_i d_{on_j} - \phi_i)$$

Derivations of equations (20) appear in Appendix A. The phase lag ϕ_i in the above partial derivatives should be deleted when $Y_i = A_i \cos\phi_i$ and $Z_i = A_i \sin\phi_i$ are taken as parameters. All of these formulae for derivatives have been checked by numerical calculation using the difference method.

Another optimization method, the so-called "DFP" or "Davidon" method, is also adopted for this study. It is a modified conjugate gradient method for minimization. The steps of iterative computations are as follows:

a. select initial vector $X^{(1)}$ for iteration and a small positive number ε for convergence criteria. If $|f_x(X^{(1)})| \le \varepsilon$, then the iterative procedure stops, otherwise proceed to step b.

b. Let $k = 1$, $G_1 = f_x(X^{(1)})$, $H_1 = I$

c. Let $Z^{(k)} = -H_k G_k$

d. Finding S_k, put $f(X^{(k)}) + S_k Z^{(k)}$ into the minimum

e. Let $X^{(k+1)} = X^{(k)} + S_k Z^{(k)}$

f. If $|f_x(X^{(k+1)})| \le \varepsilon$, the procedure stops, otherwise if $k = n$ (order of vector X), then let $X^{(1)} = X^{(n+1)}$ and return to step a, and if $k < n$, then let $G_{k+1} = f_x(X^{(k+1)})$, $\Delta X_k = X^{(k+1)} - X^{(k)}$, $\Delta G_k = G_{k+1} - G_k$

$$(21) \qquad H_{k+1} = H_k + \frac{\Delta X_k (\Delta X_k)^T}{(\Delta X_k)^T \; \Delta G_k} - \frac{H_k \Delta G_k)[(\Delta G_k)^T \; H_k]}{(\Delta G_k)^T \; H_k \Delta G_k}$$

Let $k = k + 1$ and return to step c. Here X denotes the optimized variable vector, i.e., vector L mentioned previously. Some constrained conditions are added to the system as follows:

Let $X^T = (\Delta W, \; \delta\Delta d_1, \delta\Delta d_2, \ldots, \delta\Delta d_r)$

or $X^T = (\Delta W, \delta d_{off_1}, \delta d_{off_2}, \ldots, \delta d_{off_r})$

then the constraints are:

$$W \le W_{max}$$

$$|\delta \Delta d_j| \le d_{max}$$

$$|\delta d_{off_j}| \le \text{max range of observations}$$

The method of eliminated conditions is used for processing the constrained optimization.

Let $X_j = \Delta d_j \sin Y_j$

$$W_n = \frac{1}{2}(W_{min} + W_{max}) + (W_{max} - W_{min}) \sin Y_n,$$

then X is replaced by Y, and the constrained condition can be eliminated so the problem of constrained optimization becomes an unconstrained one.

3. OPTIMUM DISTRIBUTION OF OBSERVATIONS FOR DETECTING A PERIODIC ERROR FUNCTION OF ONE PERIOD

The relative simplicity of the mathematical and optimization models for the case of a periodic error function of one period permits a rigorous analytical solution. For error functions of several periods, only numerical analyses are feasible. In this chapter we shall investigate the optimum distributions of observations for both the A,ϕ and Y,Z parameter sets and the corresponding optimum weights. The required number of measurements to meet an accuracy specification for A or ϕ and certain pathological observation distributions are covered as well.

3.1 Optimum Distribution of Observations with Respect to Parameters $Y = A\cos\phi$ and $Z = A\sin\phi$

In this case the mathematical model is given by (3). Considering (6) and (9) the trace of matrix Σ_x is as follows:

$$(22) \qquad TR(\Sigma_x) = \frac{2\sigma_o^2 x^2 \left(\sum_{j=1}^{r} \Delta S_{o_j}^2 + \sum_{j=1}^{r} \Delta C_{o_j}^2 \right)}{\sum_{j=1}^{r} \Delta S_{o_j}^2 \cdot \sum_{j=1}^{r} \Delta C_{o_j}^2 - \left(\sum_{j=1}^{r} \sin 2\theta_{om_j} \right)^2}$$

$$= \frac{2\sigma_o^2 x^2 r}{\sin^2\Delta\theta \left[r^2 - \left(\sum_{j=1}^{r} \cos 2\theta_{om_j} \right)^2 - \left(\sum_{j=1}^{r} \sin 2\theta_{om_j} \right)^2 \right]}$$

where

$$\Delta S_{o_j} = \sin \frac{2\pi}{T} d_{on_j} - \sin \frac{2\pi}{T} d_{off_j}$$

$$\Delta C_{o_j} = \cos \frac{2\pi}{T} d_{on_j} - \cos \frac{2\pi}{T} d_{off_j}$$

$$(23) \qquad \Delta\theta = \frac{1}{2} \left(\frac{2\pi}{T} (d_{on} - d_{off}) \right)$$

$$= \frac{\pi}{T} (d_{on} - d_{off}) = \frac{\pi}{T} \Delta d$$

$$\theta_{om_j} = \frac{2\pi}{T} (d_{off_j} + d_{on_j})/2 = \frac{2\pi}{T} d_{om_j}$$

In accordance with the discussion in section 2.2, the criterion for optimization is

$$TR(\Sigma_x) = \text{minimum}.$$

With regard to the distribution of observations, the numerator of (22) is constant. Suppose also that $\sin \Delta\theta$ is constant. The trace (22) will then be minimized when the denominator is maximized. This will occur when the conditions hold

$$\sum_{j=1}^{r} \cos 2\theta_{om_j} = 0$$

$$\sum_{j=1}^{r} \sin 2\theta_{om_j} = 0$$

using the identities

$$\sum_{j=1}^{r} \cos \frac{(2j-1)\pi}{r} = \sum_{j=1}^{r} \sin \frac{(2j-1)\pi}{r} = 0, \; r \neq 1,$$

gives by comparison with the above conditions

$$2\theta_{om_j} = \frac{(2j-1)\pi}{r} + 2n\pi \quad (n = 1, 2, \ldots). \tag{24}$$

Substituting according to (23) and solving for d_{omj}, the optimum distribution of the observations is

$$d_{om_j} = \frac{1}{4}(2j-1)\frac{T}{r} + \frac{nT}{2} \quad (j = 1, 2, \ldots, r). \tag{25}$$

With the optimum distribution determined, we seek a corresponding optimum calibrating weight. Again the minimum trace condition is enforced. Let the distribution related term

$$\left(\sum_{j=1}^{r} \cos 2\theta_{om_j}\right)^2 + \left(\sum_{j=1}^{r} \sin 2\theta_{om_j}\right)^2$$

be constant. The denominator now is maximized when

$$\sin^2\Delta\theta = 1,$$

from which follows

$$\Delta\theta = \frac{\pi}{2}.$$

Using (23) for $\Delta\theta$,

$$\Delta d = d_{on} - d_{off} = \frac{T}{2},$$

and the approximate relationship

$$W = \Delta d \cdot x,$$

gives the optimum calibrating weight as,

$$W = \frac{T}{2} x \ . \tag{26}$$

3.2 Optimum Distribution of Observations with Respect to Corrections to Amplitudes and Phase Lags, δA, $\delta \phi$

Using now the mathematical model (4) and the equations (5) and (9) the trace may be expressed as

$$TR(\Sigma_x) = \frac{\sigma_o^2 \, x^2 [(1+A^2)r + (A^2-1) \sum_{j=1}^{r} \cos 2\theta_{m_j}]}{A^2 \sin^2 \Delta\theta (r^2 - ((\sum_{j=1}^{r} \cos 2\theta_{m_j})^2 + (\sum_{j=1}^{r} \sin 2\theta_{m_j})^2))} \tag{27}$$

where $\theta_{m_j} = \frac{2\pi}{T} (d_{off_j} - d_{on_j})/2 - \phi$

The optimum distribution of measurements is obtained by solving the equation

$$\frac{\partial \, TR \, (\Sigma_{\delta A, \delta \phi})}{\partial_{off_j}} = 0 \ .$$

Substituting (27) into the above equation, the conditional solutions for the equation are as follows:

a. If $A < 1$

A group of θ_{m_j} whereby the conditions

$$\begin{aligned} &\sum_{j=1}^{r} \cos 2\theta_{m_j} = 1 \\ &\sin 2\theta_{m_j} = 0 \quad (j = 1, 2, \ldots, r) \end{aligned} \tag{28}$$

pertain will minimize the trace. From these conditions it follows

$$A < \sqrt{\frac{r-1}{r+1}}$$

This minimization will hold when the number of measurements is odd. When r is even, the relevant conditions become

$$\sum_{j=1}^{r} \cos 2\theta_{m_j} = 0$$

(29)

$$\sin 2\theta_{m_j} = 0 \quad (j = 1, 2, \ldots, r)$$

and it holds

$$\sqrt{\frac{r-1}{r+1}} < A < 1$$

As A becomes very much smaller than the factor $\sqrt{\frac{r-1}{r+1}}$ the criterion (28) becomes more important for the minimization of the trace.

b. If $A > 1$

Following the reasoning above, when $A > 1$ optimal observations will be those conforming to the conditions

$$\sum_{j=1}^{r} \cos 2\theta_{m_j} = -1$$

(30)

$$\sin 2\theta_{m_j} = 0 \quad (j = 1, 2, \ldots, r)$$

when r is odd and

$$A > \sqrt{\frac{r+1}{r-1}} .$$

For r even, we require

$$\sum_{j=1}^{r} \cos 2\theta_{m_j} = 0$$

(31)

$$\sin 2\theta_{m_j} = 0 \quad (j = 1, 2, \ldots, r)$$

and therefore

$$1 < A < \sqrt{\frac{r+1}{r-1}}$$

to minimize the trace. The larger A becomes, the more negligible the criterion (30) is for minimizing $TR(\Sigma_{\delta A \delta \phi})$. The criteria for distribution of observations

$$\sin 2\Theta_{m_j} = 0 \quad (j = 1, 2, \ldots, r)$$

are necessary conditions for having minimum

$$TR(\Sigma_{\delta A \delta \phi}), \quad TR(\Sigma_{\delta A}), \quad TR(\Sigma_{\delta \phi})$$

met simultaneously, because $\frac{\partial\, TR(\Sigma_{\delta A})}{\partial\, d_{off_j}}$ and $\frac{\partial\, TR(\Sigma_{\delta \phi})}{\partial\, d_{off_j}}$ have a common factor $\sin 2\Theta_{m_j}$.

With a derivation similar to the corresponding one in section 3.1, the optimum calibrating weight may once again be shown equal to $\frac{T}{2} \cdot x$.

3.3 Requisite Number of Observations for a Required Accuracy of A and ϕ

The standard deviations of δA and $\delta \phi$ can be written as:

$$m_{\delta A}^2 = \frac{\sigma_o^2 \, x^2 (r + \sum_{j=1}^{r} \cos 2\Theta_{m_j})}{\sin^2 \Delta\Theta \{ r^2 - [(\sum_{j=1}^{r} \cos 2\Theta_{m_j})^2 + (\sum_{j=1}^{r} \sin 2\Theta_{m_j})^2] \}} \tag{32}$$

$$m_{\delta \phi}^2 = \frac{\sigma_o^2 \, x^2 (r + \sum_{j=1}^{r} \cos 2\Theta_{m_j})}{A^2 \sin \Delta\Theta \{ r^2 - [(\sum_{j=1}^{r} \cos 2\Theta_{m_j})^2 + (\sum_{j=1}^{r} \sin 2\Theta_{m_j})^2] \}} . \tag{33}$$

Suppose $A \ll \sqrt{\frac{r-1}{r+1}}$,

and let $\sum_{j=1}^{r} \cos 2\Theta_{m_j} = 1$

and $\sum_{j=1}^{r} \sin 2\theta_{m_j} = 0$.

Under such assumptions (32) and (33) reduce to,

$$(34) \qquad m_{\delta A} = \frac{x}{\sin\Delta\theta \sqrt{r-1}} \sigma_o ,$$

$$(35) \qquad m_{\delta\phi} = \frac{x}{A \sin\Delta\theta \sqrt{r+1}} \sigma_o .$$

Alternatively, if

$$\sum_{j=1}^{r} \cos 2\theta_{m_j} = \sum_{j=1}^{r} \sin 2\theta_{m_j} = 0 ,$$

then under these conditions

$$(36) \qquad m_{\delta\phi} = \frac{1}{A} m_{\delta A} = \frac{x}{A \sin\Delta\theta \sqrt{r}} \sigma_o ;$$

however, $TR(\Sigma_{\delta A,\delta\phi})$ is not exactly minimized in this case. In other words, m_ϕ would decrease a little, but m_A would increase a little if $TR(\Sigma_{\delta A,\delta\phi})$ = minimum is strictly required. Such regulation also appears in the optimization for calibration of more than one wavelength (see section 4.4).

If the calibrating weight and required accuracy of A and ϕ are known, the number of observations can be determined. For instance assume the following data:

σ_o = 0.002 c.u.

$W = \frac{T}{2} x$ (x=1 mgal/c.u.)

A = 0.005 c.u.

The required accuracies are:

$m_A < 0.001$ mgal

$m_\phi < 0.087$ ($= 5°$)

Assuming optimally distributed measurements, to obtain the required accuracies, the following numbers of measurements are needed:

$r > 5$ for determining δA

$r > 20$ for determining $\delta\phi$

Eq. (35) shows that as the amplitude A becomes smaller, the determination of the phase lag ϕ becomes more difficult. On the other hand, if A is small then the effect of the error of $\delta\phi$ on the milligal equivalent will also be smaller. The maximum effect on the error of ϕ is $A\ m_\phi$. Letting $A\ m_\phi = 0.001$ mgal, the required number of observations for both the determination of A and ϕ is nearly the same. For example, assume:

$$A = 0.005 \text{ mgal}$$

$$m_A = Am_\phi = 0.001 \text{ mgal}$$

$$m_\phi = \pm\ 0.2 \text{ rad.} = \pm\ 11 \text{ degree}$$

then $r > 5$.

This is merely a basic estimate of the minimum number of measurements for the required accuracy of the adjusted A and ϕ. Below, simulative computations will show that the minimum number of observations r_{min} is approximately 45 for determining a periodic error function with 7 wavelengths if $m_A = Am_\phi < 0.001$ mgal is required. However, if m_ϕ is required to be less then 5°, the number of observations needed becomes more than 65 (see Figs. 5a-h, Section 4.5).

3.4 Inadequate Distribution of Observations

In certain cases, the distribution of measurements may prohibit the effective computation of A and ϕ. Several of such cases are presented.

A. If $d_{om} = \frac{1}{2}(d_{off} + d_{on}) \rightarrow nT/2$,

or $d_m = d_{om} - \phi \rightarrow nT/2$,

or $d_{om} \rightarrow nT/2 + T/8$,

or $d_m \rightarrow nT/2 + T/8$,

then $TR(\Sigma_x) \rightarrow \infty$,

B. If $\Delta d = d_{off_{j+1}} - d_{off_j} = \text{constant} \rightarrow nT/2$, $(n = 1, 2, \ldots)$

then $TR(\Sigma_x) \rightarrow \infty$

C. If $W \rightarrow nT$

then $TR(\Sigma_x) \rightarrow \infty$

3.5 Conclusion

The best distribution of observations for determining the unknowns $Y = A \cos\phi$ and $Z = A \sin\phi$ is a uniform distribution within the range of one period. The conditions $\sum_{j=1}^{r} \cos 2\theta_{m_j} = \sum_{j=1}^{r} \sin 2\theta_{m_j} = 0$ should be satisfied. In this case, the highest accuracy for determining the unknowns A and ϕ would not be attained, especially, when A is very small. However, the difference in accuracy is not significant.

The optimum distribution of observations for the direct determination of A and ϕ is also a uniform distribution over the range of several periods. The conditions $\sin 2\theta_{m_j} = 0$ $(j = 1, 2, \ldots r)$ are necessary for minimizing both $TR(\Sigma_{\delta A})$ and $TR(\Sigma_{\delta\phi})$.

The optimum calibrating weight is $\left(n + \frac{1}{2}\right) T \times$ $(n = 1, 2, \ldots)$ regardless of which set of parameters is selected.

The $TR(\Sigma_{\delta A})$ and $TR(\Sigma_{\delta\phi})$ are both inversely proportional to the number of observations r under the process of optimization. If the accuracy for determining A is required to be less than 0.001 mgal, the number of observations

has to be more than 5 when $\sigma_o = 0.002$ mgal. The minimum number of observations for determining ϕ is 20 to attain the RMS of ϕ less than 5°. If $m_{\delta A} = Am_{\delta\phi}$ is required, then the minimum number of observations for determining both A and ϕ is identical.

4. OPTIMIZATION FOR DETERMINING A PERIODIC ERROR FUNCTION WITH MORE THAN ONE PERIOD

4.1 Analysis of Factors to be Optimized

The condition equation (2) can be alternatively expressed as

$$(37) \qquad \Delta d_o x - 2 \sum_{j=1}^{k} \sin\left(\frac{\pi}{T_i} \Delta d_o\right) \sin\left(\frac{2\pi}{T_i} d_m - \phi_i\right) \delta A_i +$$

$$2 \sum_{j=1}^{k} A_i \sin\left(\frac{\pi}{T_i} \Delta d_o\right) \cos\left(\frac{2\pi}{T_i} d_m - \phi_i\right) \delta\phi_i + W = 0$$

where, $\Delta d_o = d_{on} - d_{off} = W/x$ (W = calibrating weight) $d_m = (d_{off} + d_{on})/2$.

The values of Δd_o and $\sin(\frac{\pi}{Ti} \Delta d_o)$ are approximately equal to the size of the calibrating weight W in each condition equation of type (37). The small value of the factor $\sin(\frac{\pi}{T_i} \Delta d_o)$ scales each coefficient of δA_i and $\delta\phi_i$ in the A matrix and the accuracy of adjusted δA_i and $\delta\phi_i$ will certainly be influenced by W. As is well known, the larger W improves the accuracy of the linear scale factor x. If x is constant over the 7000 mgal range, then only one large weight is needed for determining x with higher accuracy. If x is not constant, an additional small weight should be used for determining non-linear detail of the scale factor. In addition, a larger weight would improve the accuracy of δA_i because there would be higher correlation between x and δA_i.

The factors $\sin(\frac{2\pi}{T_i} d_m - \phi_i)$ and $\cos(\frac{2\pi}{T_i} d_m - \phi_i)$ are dependent upon the distribution of observations. The manner of distribution mainly affects the accuracy of parameters δA_i and $\delta\phi_i$, and particularly the latter. In practice an optimum distribution of observations is referred to as an adequate match between d_{m_j} (or Δd_j) and each period T_i ($i = 1, 2, \ldots k$). Consequently the general criterion for optimum distribution is that observations can be uniformly placed over one period of each wavelength. The distribution over the entire range of observations does not necessarily have to be uniform.

4.2 Optimization of Calibrating Weights

4.2.1 The optimum weight for the calibration of a periodic screw error with one fundamental frequency and second harmonic

As discussed in Section 2.2, optimum calibrating weights are found by minimizing the objective function according to $\frac{\partial \, TR(\Sigma_x)}{\partial W} = 0$. The normal equations implicit in Σ_x are formed from conditions of the type (37). Zeroes for the above minimization problem are obtained in the range 15 mgal < W < 415 mgal and are determined through the usual numerical procedures to solve a nonlinear equation. These zeroes, the optimum weights, are then used in a simulative computation to provide RMS accuracy estimates of A and ϕ. The following data, acquired from an actual bench calibration, was used for the numerical experiments:

$x = 1.048644$ mgal/c.u.

$T_1 = 70.94118$ c.u.

$T_2 = 35.47059$ c.u.

$A_1 = 0.0036639$ mgal

$A_2 = 0.004325$ mgal

$\phi_1 = -2.08027$ radians

$\phi_2 = 0.7876117$ radians

$r = 15$ observations

$d_{off_1} = 1000.5$ c.u.

interval = W/x (for uniformly distributed observations).

Two sets of zeroes corresponding to two different minima were found; the results are presented in Tables 1 and 2. A set of weights causing infinite Σ_x is presented in Table 3. Figure 1 depicts the behavior of Σ_x as a function of the weight W over the approximate range $0 < W < 300$ mgal.

Noting the ratio $W:T_1x$ from Table 1, the following relationship between the optimum calibrating weights and period T_1 can be expressed as

$$(38) \qquad W_{opt} = (k + 0.32)\, T_1 x \qquad (k = 0, 1, 2, \ldots)$$

This result may be verified analytically. Suppose that a weight W is an optimum weight for calibration provided it satisfies $\sum_{i=1}^{k} \sin \frac{\pi}{T_i} \frac{W}{x} = \sum_{i=1}^{k} \sin \frac{\pi}{T_i} W_o$ = maximum $\left(W_o = \frac{W}{x}\right)$. Based on the criterion, the formula (38) can be proved. Making a function of the form: $S(W_o) = \sin\left(\frac{\pi}{T_i} W_o\right) + \sin\left(\frac{\pi}{T_2} W_o\right)$, solve the equation: $\frac{d\,S(W_o)}{dW_o} = 0$; if $T_1 = 2T_2$, we get a quadratic equation: $4 \cos \frac{2\pi}{T_1} W_o + \cos \frac{\pi}{T_1} W_o - 2 = 0$. The solutions for the above eq. are

$$\cos \frac{\pi}{T_1} W_o = \frac{-1 \pm \sqrt{33}}{8}$$

$$\frac{W_{o1}}{T_1} = 0.3183 \qquad \text{(Taking Positive sign)}$$

$$\frac{W_{o2}}{T_1} = 0.6817 \qquad \text{(Taking negative sign)}$$

ROOT FOR $TR(\Sigma_x)$ MIN	TRACE Σ_x	$W_{opt}:T_1x$	$\sin\left(\frac{\pi}{T_i}\Delta d_o\right)$		$\frac{\Sigma_i \lvert\sin\rvert}{2}$	RMS	
			$i = 1$	$i = 2$		AMP (μgal)	PHASE (deg)
23.429	0.04458	0.3150	0.8358	0.9178	0.8768	631.6	8.553
97.821	0.04456	1.3150	-0.8358	-0.9178	0.8768	631.6	8.552
172.213	0.04456	2.3149	0.8356	0.9180	0.8768	631.5	8.552
246.605	0.04456	3.3149	-0.8356	-0.9180	0.8768	631.5	8.552
320.997	0.04456	4.3150	0.8358	0.9178	0.8768	631.5	8.552
395.389	0.04456	5.3149	-0.8358	-0.9178	0.8768	631.5	8.553

Table 1. The first set of solutions for minimum $TR(\Sigma_x)$. T_1 = 70.941 c.u., T_2 = 35.471 c.u.

ROOT FOR $TR(\Sigma_x)$ MIN	TRACE Σ_x	$W_{opt}:T_1x$	$\sin\left(\frac{\pi}{T_i}\Delta d_o\right)$		$\frac{\Sigma_i \lvert\sin\rvert}{2}$	RMS	
			$i = 1$	$i = 2$		AMP (μgal)	PHASE (deg)
50.073	0.2043	0.6731	0.8557	0.8855	0.8706	1.18	18.31
124.465	0.2043	1.6731	-0.8557	-0.8855	0.8706	1.18	18.31
198.857	0.2042	2.6731	0.8557	-0.8855	0.8706	1.18	18.31
273.249	0.2042	3.6731	-0.8557	-0.8855	0.8706	1.18	18.31
347.641	0.2042	4.6731	0.8557	-0.8855	0.8706	1.18	18.31

Table 2. The second set of solutions for minimum $TR(\Sigma_x)$. T_1 = 70.941 c.u. T_2 = 35.471 c.u.

Weight W	$\frac{\partial\ TR(\Sigma_x)}{\partial\ W}$	TRACE (Σ_x)	$W_\infty:(T_1 x)$	$\frac{\pi}{T_1}\Delta d_o$
37.199	-0.1307×10^{18}	0.2537×10^{16}	0.5000	90.0
74.475	0.1885×10^{16}	0.2030×10^{16}	1.0011	180.2
111.590	0.4792×10^{17}	0.1423×10^{17}	1.5000	270.0
148.854	-0.1423×10^{20}	0.5991×10^{18}	2.0009	0.0
185.982	-0.3460×10^{17}	0.1307×10^{17}	2.4988	89.8
223.123	-0.6283×10^{14}	0.2221×10^{17}	2.9992	179.9
260.374	-0.1122×10^{17}	0.1261×10^{17}	3.5000	270.0
297.467	-0.4317×10^{15}	0.1745×10^{17}	3.9986	359.7
334.768	0.1469×10^{16}	0.4434×10^{15}	4.5000	90.0
372.038	0.3586×10^{15}	0.1379×10^{16}	5.0010	180.2
409.159	0.1418×10^{18}	0.9198×10^{16}	5.5000	270.0

Table 3. The weights causing infinite trace (Σ_x) associated with two wavelengths $T_1 = 70.941$ c.u., $T_2 = 35.471$ c.u.

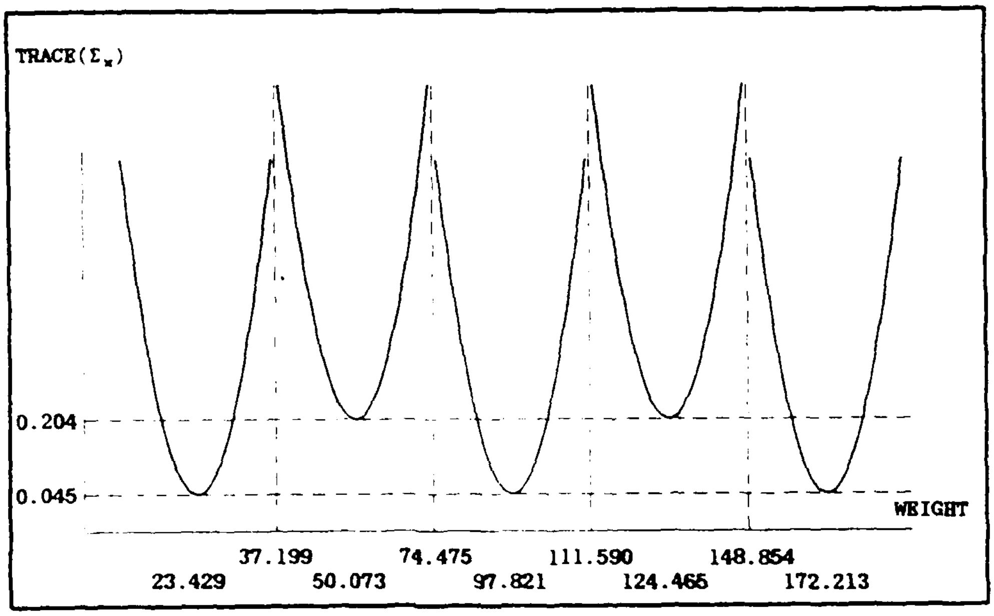


Figure 1 Relationship between trace(Σ_x) and weight W ($T_1 = 2T_2 = 70.941$ c.u.)

comparing the results with Table 2 and Table 3, the formula (38) has been proved, i.e., $W_{oi} = W_{opt}$, if we take the error of the solution for $\frac{\partial \, TR(\Sigma_x)}{\partial W} = 0$ into account.

4.2.2 The optimum weights for the calibration of several periodic screw errors

Based upon the construction of the gearbox of an LCR G gravity meter, it may happen that the periodic screw error has more than two components. In this section the optimum weights for a seven component periodic screw error are presented. As in Section 4.2.1, the equation $\frac{\partial \, TR(\Sigma_x)}{\partial W} = 0$ is first solved numerically. The roots so obtained are believed to be in error by no more than 0.001 mgal. They are then used in simulative computations to determine the RMS amplitude and phase angle data. Results are presented in Table 4.

In analyzing the results of the simulative computations we may approximately assume that it is necessary for an optimum calibrating weight W to let $\sum_{i=1}^{k} \left|\sin(\frac{\pi}{T_i} \Delta d_o)\right|$ attain maximum. Bearing in mind that $\Delta d_o = W/x$, calibrating weights which maximize this condition may be found by solving

$$\frac{\partial \sum_{j=1}^{k} \left|\sin(\frac{\pi}{T_i} \Delta d_o)\right|}{\partial \, \Delta d_o} = 0$$

Some typical results from such a computation are given in Table 5. Using this empirical evidence it may be appropriate to require the optimum weights satisfy $\sum_{j=1}^{k} \left|\sin(\frac{\pi}{T_i} \Delta d_o)\right| > 4.3$.

ROOT FOR Σ_x MIN	TRACE Σ_x	$\sin\left(\frac{\pi}{T_i}\Delta d_o\right)$								RMS	
		i = 1	i = 2	i = 3	i = 4	i = 5	i = 6	i = 7	AVE.	AMP (μgal)	PHASE (deg)
43.698	0.1916	0.0246	0.1083	0.9625	0.5220	0.7836	0.9737	0.8592	0.6048	18	9.4
54.315	0.1497	0.0306	0.1345	0.7499	0.9922	0.9752	0.4318	0.5993	0.6173	9.9	8.4
55.000	0.1176	0.0309	0.1362	0.7304	0.9978	0.8853	0.8233	0.9870	0.6558	9.6	7.4
88.540	0.1008	0.0498	0.2182	0.5626	0.9302	0.7870	0.9711	0.9178	0.6424	3.1	6.9
102.197	0.0979	0.0575	0.2511	0.9224	0.7126	0.9099	0.7548	0.9906	0.6570	2.2	6.8
121.126	0.0650	0.0681	0.2964	0.9199	0.7212	0.8854	0.8232	0.9997	0.6734	1.7	5.5
242.672	0.0580	0.1361	0.5670	0.7334	0.9971	0.9030	0.7760	0.9690	0.7253	0.9	5.2
250.350	0.2061	0.1404	0.5826	0.9118	0.7489	0.7853	0.9723	0.7356	0.6967	1.1	9.8
275.000	0.0674	0.1540	0.6312	0.8152	0.9443	0.7495	0.9924	0.6924	0.7113	0.84	5.6
324.817	0.1508	0.1817	0.7221	0.9131	0.7447	0.8027	0.9675	0.7081	0.7186	0.97	8.4
340.522	0.9580	0.1904	0.7486	0.9703	0.4696	0.5883	0.9515	0.6967	0.6593	0.74	6.7
362.077	0.1616	0.2023	0.7830	0.4054	0.7411	0.5767	0.9423	0.7728	0.6319	0.84	8.7
385.000	0.1221	0.2149	0.8171	0.5233	0.8918	0.9705	0.4683	0.4281	0.6163	0.76	7.6
399.261	0.1328	0.2227	0.8370	0.9140	0.9418	0.8143	0.9453	0.7284	0.7434	0.74	7.9

Table 4 Solutions for minimum trace Σ_x

T_1 = 5325 c.u., T_2 = 1206 c.u., T_3 = 70.941 c.u., $T_4 = \frac{1}{2} T_3$, T_5 = 7.882 c.u., $T_6 = \frac{1}{2} T_5$, T_7 = 1.0 c.u.

r = 45 observations

Δd_o (=W/x)	$\sin\left(\frac{\pi}{T_i}\Delta d_o\right)$ i = 1	2	3	4	5	6	7	SUM (3-7 only)
18.481	0.01	0.05	0.73	1.00	0.88	0.83	1.00	4.4
120.502	0.07	0.31	0.81	0.95	0.79	0.97	1.00	4.5
191.499	0.11	0.48	0.81	0.95	0.80	0.96	1.00	4.5
234.480	0.14	0.57	0.82	0.94	0.71	1.00	1.00	4.5
262.495	0.15	0.63	0.81	0.95	0.81	0.95	1.00	4.5
26.429	0.02	0.07	0.92	0.72	0.90	0.80	0.97	4.3
44.525	0.03	0.12	0.92	0.72	0.89	0.80	1.00	4.3
115.567	0.07	0.30	0.92	0.72	0.87	0.85	0.98	4.3
168.424	0.10	0.42	0.92	0.71	0.91	0.74	0.97	4.3
187.460	0.11	0.47	0.90	0.78	0.63	0.98	0.94	4.3
20.421	0.01	0.05	0.79	0.97	0.96	0.54	0.97	4.2
100.461	0.06	0.26	0.97	0.50	0.72	1.00	0.99	4.2
171.475	0.10	0.43	0.97	0.50	0.70	1.00	1.00	4.2
99.580	0.06	0.26	0.95	0.57	0.91	0.74	0.97	4.1
184.477	0.11	0.46	0.95	0.59	0.95	0.57	1.00	4.1

Table 5. Some extreme points of the function $\sum_{i=1}^{7} \sin \frac{\pi}{T_i} \Delta d_o$

4.2.3 Selection of the optimum weights and their optimum combination

As mentioned previously, one large weight and one small weight, the size of which would not be specifically required, are needed for determining the linear scale factor accurately and in detail. Likewise, two weights are needed for calibrating the circular error. In this instance the criterion

$$\sin \frac{\pi\ \Delta d_o}{T_i} = \sin \frac{\pi W}{T_i x} = \text{maximum}$$

is very important. It is significant that the value $\sin \frac{\pi \Delta d_o}{T_i}$ has an effect on the accuracy of both adjusted parameters A_i and ϕ_i. The ideal condition is for the value of all $\sin \frac{\pi \Delta d_o}{T_i}$ terms to be greater than 0.9, but it seems impossible to find one weight which satisfies the conditions for all wavelengths. Therefore, we have to select more than one weight for calibration. Some of the optimum weights from the above section are adopted in the simulative computation for further comparison and optimizing.

Numerical experiments demonstrate that the accuracies of $\delta\ _i$ and $\delta\phi_i$ are very sensitive to the value $\sin \frac{\pi \Delta d_o}{T_i}$. Table 6 shows how the change in the value of $\sin \frac{\pi \Delta d_o}{T_i}$ causes a change in the standard deviation of the parameters δA_i and especially $\delta\phi_i$. If

$$\left| \sin \frac{\pi \Delta d_o}{T_i} \right| > \left| \sin \frac{\pi \Delta d_o}{T_j} \right| ,$$

then $m_{\delta A_i}$ or $m_{\delta\phi_i}$ is certainly less than $m_{\delta A_j}$ or $m_{\delta\phi_j}$, respectively. $A_3 = A_4 = A_5 = A_6 = A_7$ has been assumed in the computation to avoid obscuring the play of weight because the amplitude also significantly influences the accuracy of $\delta\phi_i$. Of particular interest are the deficient values for $\sin \frac{\pi \Delta d_o}{T_i}$ ($i = 4, 5, 7$) for the 180 mgal weight, as evidenced by the large RMS $\delta\phi_i$ values. Similar results occurred for $i = 3, 6$ in the 242.672 mgal weight.

The results of the simulative computation of the adjustment in which some combinations of weights are used are listed in Table 8. They show that the deficient value of $\sin \frac{\pi \Delta d_o}{T_i}$ associated with one weight could be compensated by the other so higher and homogeneous accuracy of adjusted δA_i and $\delta\phi_i$ could be obtained. It is obvious that the value of RMS δA or RMS $\delta\phi$ exactly follows the change in the value of $\sin \frac{\pi \Delta d_o}{T_i}$, as does the value of RMS Y or RMS Z.

A. r = 30 observations

Period c.u.	$\sin \frac{\pi W_1}{T_i x}$	RMS δA μgal	RMS $\delta\phi$ deg	$\sin \frac{\pi W_2}{T_i x}$	RMS δA μgal	RMS $\delta\phi$ deg
5325	0.10	3.6	0.72	0.14	2.8	0.53
1206	0.43	1.1	6.32	0.57	0.7	3.86
70.941	0.97	0.4	5.30	0.73	0.6	7.74
35.471	0.48	1.0	12.30	1.00	0.4	4.55
7.882	0.65	1.6	7.88	0.90	0.4	5.49
3.941	0.99	0.4	5.00	0.78	0.5	7.40
1.0	0.89	0.4	5.63	0.96	0.4	5.23
Ave.	0.64			0.73		
RMS		1.5	6.96		1.2	5.45

B. r = 66 observations

Period c.u.	$\sin \frac{\pi W_1}{T_i x}$	RMS δA μgal	RMS $\delta\phi$ deg	$\sin \frac{\pi W_2}{T_i x}$	RMS δA μgal	RMS $\delta\phi$ deg
5325	0.10	2.5	0.51	0.14	1.9	0.34
1206	0.43	0.6	3.50	0.57	0.4	2.60
70.941	0.97	0.3	3.83	0.73	0.3	4.63
35.471	0.48	0.5	6.99	1.00	0.3	3.27
7.882	0.65	0.4	5.15	0.90	0.3	3.65
3.941	0.99	0.3	3.15	0.78	0.3	4.36
1.0	0.89	0.3	3.80	0.96	0.3	3.44
Ave.	0.64			0.73		
RMS		1.0	4.30		0.8	3.45

C. r = 90 observations

Period c.u.	$\sin \frac{\pi W_1}{T_i x}$	RMS δA μgal	RMS $\delta\phi$ deg	$\sin \frac{\pi W_2}{T_i x}$	RMS δA μgal	RMS $\delta\phi$ deg
5325	0.10	2.1	0.42	0.14	1.6	0.30
1206	0.43	0.5	2.99	0.57	0.4	2.22
70.941	0.97	0.2	3.02	0.73	0.3	3.36
35.471	0.48	0.5	6.02	1.00	0.2	2.76
7.882	0.65	0.3	4.39	0.90	0.2	3.13
3.941	0.99	0.2	2.90	0.78	0.3	3.63
1.0	0.89	0.2	3.22	0.96	0.2	2.96
AVE.	0.64			0.73		
RMS		0.8	3.64		0.7	2.91

W_1 = 180 mgal, W_2 = 242.672 mgal
A_1 = 0.2957 mgal, A_2 = 0.0095 mgal
$A_3 = A_4 = A_5 = A_6 = A_7$ = 0.0043 mgal

Table 5. The Relationship Between the Accuracies of Adjusted Parameters and the Value of $\sin \frac{\pi W}{T_i x}$ According to the Variant Number of Measurements

A. W_1 = 242.672 mgal, W_2 = 180.0 mgal

			r_1 = 50 obs. r_2 = 16		r_1 = 50 obs. r_2 = 40	
Period c.u.	$\sin \frac{\pi W_1}{T_i x}$	$\sin \frac{\pi W_2}{T_i x}$	RMS δA μgal	RMS $\delta\phi$ deg	RMS δA μgal	RMS $\delta\phi$ deg
5325	0.14	0.10	2.0	0.38	1.8	0.34
1206	0.57	0.43	0.5	2.78	0.4	2.48
70.941	0.73	0.97	0.4	4.94	0.3	3.50
35.471	1.00	0.48	0.3	3.61	0.3	3.43
7.882	0.90	0.65	0.3	3.91	0.3	3.54
3.941	0.78	0.99	0.4	4.59	0.3	3.41
1.0	0.96	0.89	0.3	3.64	0.2	3.12
Ave.	0.73	0.64				
RMS			0.8	3.68	0.7	3.03

B. W_1 = 242.672 mgal, W_2 = 88.540 mgal

			r_1 = 50 obs. r_2 = 16		r_1 = 50 obs. r_2 = 40	
Period c.u.	$\sin \frac{\pi W_1}{T_i x}$	$\sin \frac{\pi W_2}{T_i x}$	RMS δA μgal	RMS $\delta\phi$ deg	RMS δA μgal	RMS $\delta\phi$ deg
5325	0.14	0.05	2.1	0.39	2.0	0.38
1206	0.57	0.22	0.5	2.91	0.5	2.81
70.941	0.73	0.56	0.4	4.77	0.3	4.28
35.471	1.00	0.93	0.3	3.28	0.2	2.82
7.882	0.90	0.79	0.3	3.78	0.3	3.28
3.941	0.78	0.97	0.3	4.25	0.3	3.50
1.0	0.96	0.98	0.3	3.66	0.2	3.18
AVE.	0.73	0.64				
RMS			0.8	3.54	0.8	3.10

C. W_1 = 242.672 mgal, W_2 = 19.393 mgal

			r_1 = 50 obs. r_2 = 16		r_1 = 50 obs. r_2 = 40	
Period c.u.	$\sin \frac{\pi W_1}{T_i x}$	$\sin \frac{\pi W_2}{T_i x}$	RMS δA μgal	RMS $\delta\phi$ deg	RMS δA μgal	RMS $\delta\phi$ deg
5325	0.14	0.01	2.1	0.40	2.1	0.40
1206	0.57	0.05	0.5	2.97	0.5	2.96
70.941	0.73	0.73	0.3	4.56	0.3	3.88
35.471	1.00	1.00	0.3	3.38	0.2	2.90
7.882	0.90	0.88	0.3	3.65	0.2	3.13
3.941	0.78	0.82	0.3	4.16	0.3	3.50
1.0	0.96	1.00	0.3	3.32	0.3	2.76
AVE.	0.73	0.64				
RMS			0.9	3.44	0.8	2.98

A_1 = 0.2957 mgal, A_2 = 0.0095 mgal

$A_3 = A_4 = A_5 = A_6 = A_7$ = 0.0043 mgal

Table 7. The Results of Simulative Adjustment for Estimating and Comparing the Accuracies of Adjusted Parameters According to the Variant Combination of Two Weights in the Calibration.

A. $W_1 = 180.0$ mgal, $W_2 = 38.540$ mgal

Period c.u.	$\sin \frac{\pi W_1}{T_i x}$	$\sin \frac{\pi W_2}{T_i x}$	$r_1 = 50$ obs., $r_2 = 16$ RMS δA µgal	RMS $\delta\phi$ deg	$r_1 = 50$ obs., $r_2 = 40$ RMS δA µgal	RMS $\delta\phi$ deg
5325	0.10	0.05	2.7	0.58	2.5	0.53
1206	0.43	0.22	0.6	3.76	0.6	3.54
70.941	0.97	0.56	0.3	3.95	0.3	3.63
35.471	0.48	0.93	0.4	5.38	0.3	3.92
7.882	0.65	0.79	0.4	4.89	0.3	3.96
3.941	0.99	0.97	0.3	3.63	0.2	3.16
1.0	0.89	0.98	0.3	3.74	0.2	3.27
AVE.	0.64	0.64				
			1.1	3.97	1.0	3.33

B. $W_1 = 180.0$ mgal, $W_2 = 20.0$ mgal

Period c.u.	$\sin \frac{\pi W_1}{T_i x}$	$\sin \frac{\pi W_2}{T_i x}$	$r_1 = 50$ obs., $r_2 = 16$ RMS δA µgal	RMS $\delta\phi$ deg	$r_1 = 50$ obs., $r_2 = 40$ RMS δA µgal	RMS $\delta\phi$ deg
5325	0.10	0.01	2.7	0.60	2.7	0.60
1206	0.43	0.05	0.7	3.90	0.7	3.88
70.941	0.97	0.75	0.3	3.81	0.2	3.36
35.471	0.48	0.99	0.4	5.37	0.3	3.76
7.882	0.65	0.97	0.3	4.61	0.3	3.52
3.941	0.99	0.48	0.3	3.77	0.3	3.55
1.0	0.89	0.22	0.3	4.27	0.3	4.19
AVE.	0.64	0.50				
RMS			1.1	4.01	1.1	3.45

C. $W_1 = 180.0$ mgal, $W_2 = 19.393$ mgal

Period c.u.	$\sin \frac{\pi W_1}{T_i x}$	$\sin \frac{\pi W_2}{T_i x}$	$r_1 = 50$ obs., $r_2 = 16$ RMS δA µgal	RMS $\delta\phi$ deg	$r_1 = 50$ obs., $r_2 = 40$ RMS δA µgal	RMS $\delta\phi$ deg
5325	0.10	0.01	2.7	0.60	2.7	0.60
1206	0.43	0.05	0.7	3.91	0.7	3.88
70.941	0.97	0.73	0.3	3.87	0.2	3.38
35.471	0.48	1.00	0.4	5.83	0.3	4.06
7.882	0.65	0.88	0.3	4.71	0.3	3.72
3.941	0.99	0.82	0.3	3.61	0.2	3.11
1.0	0.89	1.00	0.3	3.76	0.3	2.87
AVE.	0.64	0.64				
RMS			1.1	4.04	1.1	3.28

$A_1 = 0.2957$ mgal, $A_2 = 0.0095$ mgal

$A_3 = A_4 = A_5 = A_6 = A_7 = 0.0043$ mgal

Table 8. The Results of Simulative Adjustment for Estimating and Comparing the Accuracies of Adjusted Parameters According to the Variant Combination of Two Weights in the Calibration.

Concerning the number of observations, it is reasonable to distribute more observations for using the weight which has the larger value of the sum of $\sin \frac{\pi \Delta d_o}{T_i}$ (see Tables 7, 8). One large weight is necessary for improving the accuracy of the parameters of the long periodic screw error, and the weight could also be used in precisely determining short periodic screw errors provided it has a large value for $\sin \frac{\pi \Delta d_o}{T_i}$ for each short wavelength. If non-linear scale factor detail is not taken into account, it would be better to select two large weights which could be compensated with each other in the value of $\sin \frac{\pi \Delta d_o}{T_i}$.

The sufficient compensation of the defective value of $\sin \frac{\pi \Delta d_o}{T_i}$ with the other should be considered as a principle for selecting an optimum combination of calibrating weights. We suggest that the combination of weights should be selected by the following criteria:

$$(39) \qquad \sum_{i=3}^{7} \left| \sin \frac{\pi W_1}{T_i x} \right| / 7 \geq 0.88$$

$$(40) \qquad \left|\sin \frac{\pi W_1}{T_i x}\right| + \left|\sin \frac{\pi W_2}{T_i x}\right| \geq 1.6$$
$$(i = 3, 4, \ldots 7)$$

where W_1 is a large weight and W_2 a compensative weight. According to these criteria and Table 5, several possible combinations of weights are selected in Table 9. Note that the weight of option depends on the scale factor x, because $W = \Delta d_o\, x$. There are some differences in the scale factor x between different gravity meters; therefore, no universal weight can be fitted for all models. If a change in the value of x is assumed to be 0.005, then the corresponding variation of weight W would be from about 0.1 to 1.0 mgal. The

Δd_o (For large weight)	$\sin \frac{\pi \Delta d_o}{T_1}$	$\sin \frac{\pi \Delta d_o}{T_2}$	$\sin \frac{\pi \Delta d_o}{T_3}$	$\sin \frac{\pi \Delta d_o}{T_4}$	$\sin \frac{\pi \Delta d_o}{T_5}$	$\sin \frac{\pi \Delta d_o}{T_6}$	$\sin \frac{\pi \Delta d_o}{T_7}$	|AVERAGE| $(T_3 - T_7)$
191.499	0.11	0.48	0.81	0.95	0.80	0.96	1.00	0.90
$W = \Delta d_o \cdot x$ (= 200.82)								
234.480	0.14	0.57	0.82	0.94	0.71	1.00	1.00	0.89
(245.886)								
262.495	0.15	0.63	0.81	0.95	0.81	0.95	1.00	0.90
(275.264)								
For compensative W								
26.429	0.02	0.07	0.92	0.72	0.90	0.80	0.97	0.86
(27.715)								
44.525	0.03	0.12	0.92	0.72	0.89	0.80	1.00	0.87
(46.691)								
115.524	0.07	0.30	0.92	0.72	0.88	0.83	1.00	0.87
(121.144)								
168.424)	0.10	0.42	0.92	0.71	0.91	0.74	0.97	0.85
(176.617)								

Table 9. The weights for the best combination

corresponding change in the value of $\sin \frac{\pi \Delta d_o}{T_i}$ can range from $0.15/T_i$ to $1.5/T_i$. It is obvious that the change will have a significant effect on the accuracy of determining δA and $\delta\phi$.

We suggest that every gravity meter should have its own special calibrating weights which fit with its scale factor and periods of circular errors. The special calibrating weights should be considered as an accessory and supplied with gravity meters to the users.

4.3 The Optimum Intervals for Uniform Distribution of Observations

4.3.1 Four illustrative numerical examples

It is very important for a uniform distribution to have an adequate interval of observations because of the periodicity of the circular error function. We should carefully arrange the interval for making uniformly distributed measurements. Otherwise, the results would be incorrect, regardless of how many observations are made. In this section the results of four numerical examples are presented in the following pages to illustrate some important points. The observations which were used for the experiments may be found in Appendix C.

Experiment 1

r = number of observations = 30

σ_0 = a priori variance of observations = 0.0018 c.u.

W = calibrating weight = 180.0 mgal

Δd = 180 counter units

Adjusted Parameters

Scale: x = 1.048 mgal/c.u.
A posteriori standard deviation of scale: $\hat{\sigma}_x$ = 0.0005 mgal/c.u.

i	Amplitude (μgal)	Phase (deg)	Y = A cos ϕ (μgal)	$\hat{\sigma}_Y$	Z = A sin ϕ (μgal)	$\hat{\sigma}_Z$
1	246.83	22.4	228.3	96.4	93.9	40.1
2	9.00	238.3	-4.7	1.5	-7.7	3.7
3	2.92	254.3	-0.8	0.6	-2.8	1.2
4	4.96	70.3	1.7	1.5	4.7	1.3
5	1.33	342.1	1.3	1.4	-0.4	0.9
6	2.60	288.0	0.8	0.4	-2.5	1.2
7	44.72	193.5	-43.5	60.5	-10.4	33.3

Matrix of correlations for Amplitudes
(standard deviations along diagonal in μgal)

1	104.	0.95	0.96	0.82	0.85	0.96	0.38
2		3.79	0.91	0.73	0.71	0.92	0.26
3			1.32	0.80	0.82	0.93	0.38
4				1.69	0.72	0.78	0.66
5	symmetric				1.39	0.80	0.33
6						1.19	0.36
7							28.6

RMS standard deviation - Amplitude: 41 μgal

Matrix of correlations for Phase angles
(standard deviations along diagonal in degrees)

1	0.70	-0.24	-0.93	0.38	-0.28	0.39	-0.29
2		6.88	0.81	-0.23	0.55	-0.82	0.19
3			5.72	-0.20	0.21	-0.11	0.32
4				13.27	-0.45	0.11	-0.54
5	symmetric				22.37	-0.12	0.49
6						5.70	-0.25
7							809.34

RMS standard deviation - Phase: 310°

Experiment 2

$r = 30$

$\sigma_o = 0.0018$ mgal

$W = 180.0$ mgal

$\Delta d = 171.67$ c.u.

Adjusted Parameters

Scale $x = 1.048$ mgal/c.u.
$\hat{\sigma}_x = 0.00316$ mgal/c.u.

i	Amplitude (μgal)	Phase (deg)	$Y = A \cos \phi$ (μgal)	$\hat{\sigma}_Y$	$Z = A \sin \phi$ (μgal)	$\hat{\sigma}_Z$
1	295.72	21.7	276.6	3.7	110.2	3.8
2	9.50	242.6	-4.4	1.1	-8.5	1.1
3	3.66	246.2	-1.5	0.4	-3.5	0.4
4	4.32	41.8	3.2	1.0	2.9	1.0
5	2.32	1.4	2.6	0.6	0.1	0.6
6	3.56	300.6	1.5	0.4	-2.5	0.4
7	4.45	257.5	-0.9	0.4	-3.9	0.4

Matrix of correlations - Amplitude
(standard deviations along diagonal in μgal)

1	3.68	-0.02	0.01	-0.02	0.01	-0.01	0.00
2		1.09	-0.09	0.50	-0.19	0.08	-0.06
3			0.41	-0.05	-0.05	-0.20	0.03
4				0.96	-0.15	0.04	-0.06
5					0.60	-0.03	0.03
6						0.39	0.01
7							0.43

RMS standard deviation - Amplitude: 1.5 μgal

Matrix of correlations - Phase
(standard of deviations along diagonal in degrees)

1	0.75	-0.02	0.04	-0.06	0.10	0.05	0.09
2		6.67	0.02	-0.49	0.19	0.04	0.10
3			6.19	-0.07	0.05	0.19	0.01
4				13.3	-0.24	-0.09	-0.17
5					15.3	0.06	0.13
6						6.38	0.02
7							5.65

RMS standard deviation - Phase: 9°

Experiment 3

$r = 30$

$W = 203.894$ mgal

$\Delta d = 194.436$ c.u.

Standard deviations of adjusted parameters

RMS $\delta x = 0.003$ μgal/c.u.

RMS	T_1	T_2	T_3	T_4	T_5	T_6	T_7	Mean RMS
δA (μgal)	3.33	0.77	0.54	0.39	0.60	0.55	0.41	0.34
$\delta\phi$ (deg)	0.62	4.85	6.79	5.21	72.57	79.56	5.56	40.93
Y (μgal)	3.29	0.78	0.51	0.39	6.03	6.01	0.41	3.48
Z (μgal)	3.23	0.79	0.54	0.39	5.45	5.45	0.41	3.19

Experiment 4

$r_1 = 15$, $r_2 = 15$

$W_1 = 203.894$ mgal, $W_2 = 46.691$ mgal

$\Delta d_1 = 194.436$ c.u., $\Delta d_2 = 44.525$ c.u.

Standard deviations of adjusted parameters

RMS $\delta x = 0.008$ μgal/c.u.

RMS	T_1	T_2	T_3	T_4	T_5	T_6	T_7	Mean RMS
δA (μgal)	9.67	1.13	0.55	0.41	0.55	0.50	0.42	3.70
δφ (deg)	0.85	6.62	6.96	7.50	7.39	6.66	5.03	6.25
Y (μgal)	8.95	1.11	0.52	0.41	0.55	0.50	0.37	3.43
Z (μgal)	5.72	1.12	0.55	0.56	0.56	0.49	0.43	2.25

Let us now analyze the results of the four experiments. In Section 3.4 we saw that the standard deviation of adjusted parameters will tend to infinity if $\Delta d = nT_i$ (n being a positive integer), i.e. the interval of observations is equal to an integral multiple of period T_i. In experiment 1, the standard deviation of $\delta A_7 = 28.6$ μgal and the standard deviation of $\delta\phi_7 = 309$ degrees are large because $\Delta d = 180\ T_7$. In experiment 2, $\Delta d = W/x = 171.67$ is not equal to any integral multiple of any period T_i, so the accuracy is immediately improved.

There are many inadequate intervals Δd_∞ which are approximately equal to nT_i. In this case it is easy to understand the reason why the intervals $\Delta d = nT_i$ are inadequate. However, there are also other inadequate intervals which cannot be expressed by $\Delta d = nT_i$. In experiment 3, $\Delta d = W/x = 194.436$ is an inadequate interval. The relationships between Δd and T_i are as follows:

$$\begin{aligned}\Delta d &= 0.04\ T_1\\ &= 0.16\ T_2\\ &= 2.74\ T_3\\ &= 5.48\ T_4\\ &= 24.67\ T_5\\ &= 49.34\ T_6\\ &= 194.44\ T_7\end{aligned}$$

Considering multiples of Δd, however, it becomes evident

$$3\ \Delta d = 74\ T_5 = 148\ T_6$$

This expression indicates that Δd and T_5 have no common factor except one, and the observations d_m ($= \frac{1}{2}(d_{on} + d_{off})$) are uniformly placed only on three different positions over periods T_5 or T_6, no matter how many observations are made. Table 10 shows the distribution of the 30 observations used for

obs #	α_5 $W_1 = 203.894$ $\Delta d = 194.436$	α_6 $W_1 = 203.894$ $\Delta d = 194.436$	obs #	α_5 $W_2 = 46.691$ $\Delta d = 44.525$		α_6 $W_2 = 46.691$ $\Delta d = 44.525$
1	0.0021	0.0044	16	0.5331		0.0662
2	0.3387	0.6774	17	0.8279		0.6558
3	0.6785	0.3498	18	0.1254		0.2508
4	0.0126	0.0252	19	0.4247		0.8494
5	0.3494	0.6990	20	0.7219		0.4438
6	0.6841	0.3683	21	0.0169		0.0338
7	0.0157	0.0316	22	0.3154		0.6308
8	0.3554	0.7110	23	0.6149		0.2298
9	0.6843	0.3688	24	0.9099		0.8198
10	0.0216	0.0432	25	0.2068		0.4136
11	0.3551	0.7104	26	0.5053		0.0106
12	0.6906	0.3813	27	0.8045		0.6090
13	0.0185	0.0371	28	0.0974		0.1948
14	0.3529	0.7061	29	0.3988		0.7976
15	0.6853	0.3709	30	0.6842		0.3884
16	0.0177	0.0356				
17	0.3538	0.7078	21	0.0169	26	0.0106
18	0.6837	0.3677	28	0.0974	21	0.0338
19	0.0181	0.0365	18	0.1254	16	0.0662
20	0.3515	0.7033	25	0.2068	28	0.1948
21	0.6837	0.3677	22	0.3154	23	0.2298
22	0.0183	0.0369	29	0.3988	18	0.2508
23	0.3570	0.7144	19	0.4247	30	0.3884
24	0.6916	0.3836	26	0.5053	25	0.4136
25	0.0243	0.0489	16	0.5331	20	0.4438
26	0.3619	0.7241	23	0.6149	27	0.6090
27	0.6975	0.3953	30	0.6942	22	0.6308
28	0.0325	0.0655	20	0.7219	17	0.6558
29	0.3726	0.7455	27	0.8045	29	0.7976
30	0.7063	0.4131	17	0.8279	24	0.8198
			24	0.9099	19	0.8494

Table 10. An example of the phase distribution over a period produced by one adequate interval of observations and one inadequate interval of observations

α_i = decimal part of $2d_m/T_i$

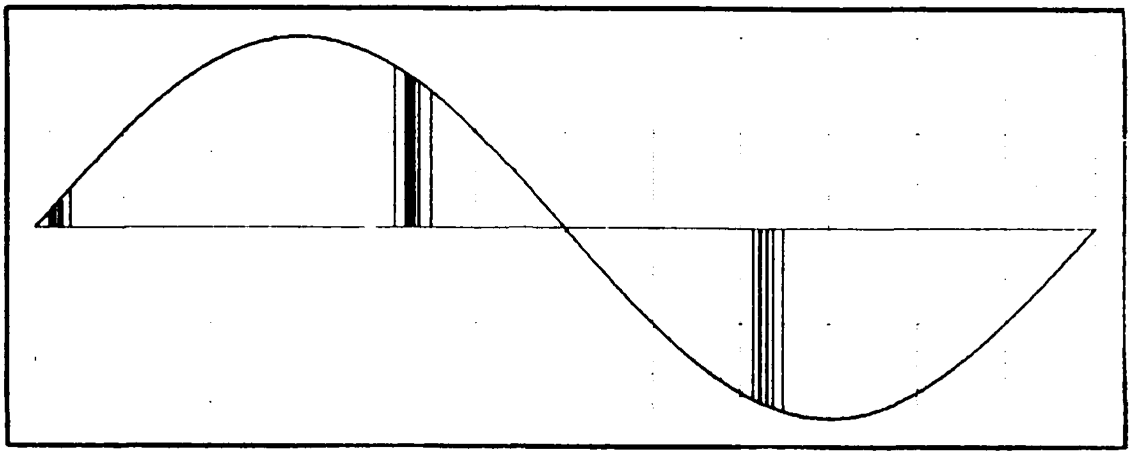
experiment 3 over periods T_5 and T_6 (columns 2 and 3). It is obvious that all 30 observations are approximately equivalent to only 3 observations to be used for solving the four parameters δA_5, δA_6, $\delta\phi_5$, and $\delta\phi_6$.

In experiment 4, 15 observations with intervals $\Delta d_2 = W_2/x = 44.525$ replace the sixteenth to thirtieth observations used in experiment 3. These 15 new measurements were made with $W_2 = 46.691$ mgal. The relationship between Δd_2 and T_5 is $15\ \Delta d_2 = 84.73\ T_5$. Table 10 shows that the 15 observations are uniformly distributed at 15 different points over periods T_5 and T_6 respectively, giving in total 18 locations of observations over T_5 and T_6. The accuracy of the results by using both of Δd_1 and Δd_2 may be seen to be much better than by using only Δd_1.

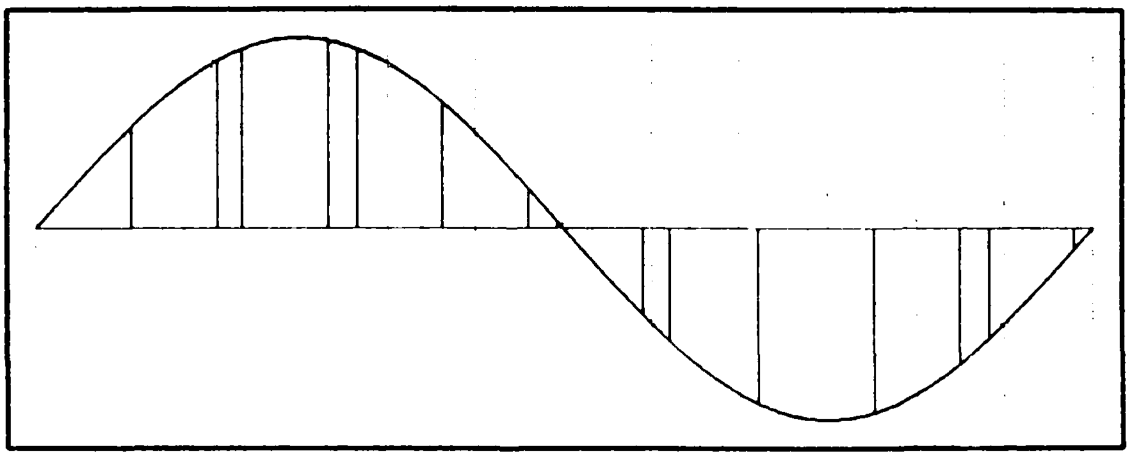
In Figure 2 the phase distributions of observations over period T_5 corresponding to intervals $\Delta d_1 = 194.436$ and Δd_2 44.525 are shown. From these figures it is clear that the superior solution for the wave's parameters would be the one using measurements distributed in conformance with Δd_2.

4.3.2 Solutions for the equation $\dfrac{\partial\ TR(\Sigma_x)}{\partial\ \Delta d} = 0$

The numerical examples of Section 4.3.1 served to provide some preliminary insight into the relationship between the distribution of observations and the accuracies of the adjusted parameters. It is our goal to determine which is the optimal uniform distribution of measurements. To comply with our specified optimization criterion we attempt to determine such a Δd whereby $TR(\Sigma_x)$ = minimum. These Δd values have been computed numerically using (13) and the partial derivative expressions (20). Considering the numerical procedures used, the computed roots are believed to be in error by less than 0.001 c.u. Several sets of solutions to the equation $\dfrac{\partial\ TR(\Sigma_x)}{\partial \Delta d} = 0$



2a. Δd_1 = 194.436 c.u. r_1 = 30 observations



2b. Δd_2 = 44.525 r_2 = 15

Figure 2. The distribution of observations related to period T_5 = 7.882 c.u.

are presented in Tables 11a,b. Notice each table pertains to a different periodic screw error function comprised of varying combinations of periods.

From Table 11a it may be seen that $TR(\Sigma_x)$ attains a minimum for Δd = 29.903 c.u. This should be considered the optimum value for the particular periodic screw error function identified below the table. Table 11a also illustrates another factor to consider. Finding extreme points of a function using the calculus may give a Δd_∞ for which $TR(\Sigma_x)$ is maximum. As shown in Table 11b $TR(\Sigma_x)$ has a minimum at Δd = 162.842 c.u. for the seven component screw error function

Further information about the relationship between the interval Δd and the resulting minimum or maximum trace of Σ_x appears in Table 12. Some values for intervals Δd_{opt} and Δd_∞ were obtained from the computation whose results appeared in Table 11b. Table 12 shows that although 45 observations were made, the equivalent observations corresponding to Δd_∞ are much less than the practical ones; in fact, often even less than the minimum number of observations required for determination of unknowns. With optimum intervals of observations, there are almost no repeated positions of observation over each period T_i. All observations are uniformly or nonuniformly distributed over each period involved. For example, a group of observations with Δd_{opt} = 162.842 are uniformly placed over each period.

In a more general setting, the relationship between interval Δd and period T_i may be written in the form:

(41) $$n\Delta d = m_i T_i$$

or $$\frac{\Delta d}{T_i} = \frac{m_i}{n} \qquad (i = 1, 2, \ldots, k)$$

ROOTS FOR OPTIMUM Δd	$TR(\Sigma_x)$	RMS AMPLITUDE (μgal)	RMS PHASE (deg)
6.590	0.2992	1.3	18.09
9.712	0.2956	1.3	17.98
10.904	0.2929	1.3	17.90
19.356	0.2979	1.3	18.06
22.853	0.2988	1.3	18.08
26.720	0.3097	1.3	18.41
27.784	0.2967	1.3	18.01
29.903	0.2907	1.3	17.83
35.268	0.3131	1.2	18.51
The following Δd give maximum $TR(\Sigma_x)$			
4.437	4.0×10^2	49.0	659.71
6.700	1.8×10^2	33.2	445.0
8.500	8.0×10^2	34.8	435.0
15.700	2.9×10^4	129.5	5583.8
22.900	7.2	6.8	88.48
24.040	8.4	14.8	96.06
30.923	5.0×10^2	52.9	740.2
31.900	6.7	8.6	83.53
35.500	2.7×10^8	18129.5	5.4×10^5

Table 11a. Some solutions for minimum and maximum $TR(\Sigma_x)$. The periodic screw error function has the three periods

$T_1 = 7.882$ c.u., $T_2 = 3.941$ c.u., $T_3 = 1.0$ c.u.

$W = 20.0$ mgal, $r = 21$ observations

$\Delta d_{opt} = 29.903$, $TR(\Sigma_x)$ = minimum

ROOTS FOR OPTIMUM Δd	$TR(\Sigma_x)$	RMS AMPLITUDE (μgal)	RMS PHASE (deg)
52.150	0.055	3.1	5.07
57.102	0.057	2.9	5.15
58.851	0.057	2.8	5.16
65.572	0.056	2.6	5.10
67.633	0.075	2.4	5.92
68.128	0.062	2.6	5.40
74.217	0.059	1.9	5.26
77.129	0.056	1.8	5.11
77.600	0.057	1.8	5.19
80.576	0.055	1.6	5.07
85.038	0.063	1.8	5.43
89.877	0.055	1.4	5.06
93.218	0.056	1.3	5.10
96.335	0.055	1.3	5.07
105.151	0.061	1.3	5.34
114.442	0.058	1.3	5.21
128.854	0.055	1.2	5.06
133.328	0.064	1.3	5.48
136.061	0.054	1.3	5.07
153.278	0.056	1.3	5.11
162.842	0.054	1.2	5.04
164.663	0.060	1.2	5.28
166.396	0.056	1.2	5.10
167.065	0.056	1.2	5.10
168.669	0.054	1.2	5.05
158.350	0.054	1.2	5.04

Table 11b. Some solutions for minimum $TR(\Sigma_x)$. The periodic screw error function has the seven periods T_1 = 5325 c.u., T_2 = 1206. c.u., T_3 = 70.941 c.u., T_4 = 35.471 c.u., T_5 = 7.882 c.u., T_6 = 3.941 c.u., T_7 = 1.0 c.u.
W = 130 mgal, r = 45 observations

Δd_{opt}	RELATIONSHIP BETWEEN Δd AND T_i	Δd_{∞}	RELATIONSHIP BETWEEN Δd AND T_i
52.150	$23\ \Delta d = 0.99\ T_2$	83.84	$11\ \Delta d = 13\ T_3$
	$= 16.91\ T_3$		$= 117\ T_5$
	$= 152.17\ T_5$		
57.102	$44\ \Delta d = 2.08\ T_2$	91.211	$7\ \Delta d = 9\ T_3$
	$= 35.42\ T_3$		$= 81\ T_5$
	$= 318.75\ T_5$		
58.851	$44\ \Delta d = 2.15\ T_2$	94.588	$3\ \Delta d = 4\ T_3$
	$= 36.50\ T_3$		$= 36\ T_5$
	$= 73.00\ T_4$		
	$= 328.51\ T_5$		
65.572	$53\ \Delta d = 2.88\ T_2$	97.545	$8\ \Delta d = 11\ T_3$
	$= 48.99\ T_3$		$= 99\ T_5$
	$= 440.40\ T_5$		
67.633	$43\ \Delta d = 2.41\ T_2$	102.350	$9\ \Delta d = 13\ T_3$
	$= 40.99\ T_3$		$= 117\ T_5$
	$= 368.95\ T_5$		
68.128	$53\ \Delta d = 2.99\ T_2$	107.131	$2\ \Delta d = 3\ T_3$
	$= 50.90\ T_3$		$= 27\ T_5$
	$= 458.08\ T_5$		
74.217	$65\ \Delta d = 4.00\ T_2$	106.350	$2\ \Delta d = 3\ T_3$
	$= 68.00\ T_3$		$= 27\ T_5$
	$= 612.00\ T_5$		

Table 12. The relationship between Δd_{opt} (or Δd_{∞}) found by solving $\partial TR(\Sigma_x)/\partial \Delta d = 0$ and T_i ($i = 1, 2, \ldots 7$)

n and m_i are mutual prime numbers, i.e., n and m_i have no common factor other than one. The sizes of n and m_i will influence whether the interval Δd will result in a minimum or maximum $TR(\Sigma_x)$.

From this discussion it follows that two sets of observations could be made with the same weight but different interval Δd. At a first glance they may appear as two completely different sets of observations, with different distributions over the range of counter readings. However, the distribution of their phases corresponding to both sets of observations over the range of any period involved may be identical. This would consequently result in the same standard deviation of adjusted parameters for both sets of observations.

4.3.3 The effective observations and the optimum effective observations

We saw in Section 4.3.2 that the accuracy of the parameters depends not on the distribution of observations over the range of counter readings. Rather, it depends on the distribution of the phases over the range of each period involved, provided the other conditions for making observations, such as the calibrating weights, are the same. The number n in (41) defines the number of observations which occupy n separate phases over period T_i. The n equivalent observations under the uniform distribution are "effective" for determination of unknowns. Concerning this, the following definitions may be established:

<u>The effective observations</u>

If the relationship between the interval for a uniform distribution of observations Δd and the period T_i can be expressed by the form (41), the number n in the expression is referred to as the number of effective observations for the period T_i.

A perfect group of effective observations

If the number of the effective observations are equal to the practical number of observations, namely $n = r$, a perfect group of effective observations for the period T_i is made.

The optimum group of effective observations

The perfect group of effective observations for all of the periods T_i ($i = 1, 2, \ldots, k$) is referred to as the optimum group of effective observations, and corresponding intervals are the optimum ones.

A group of observations with uniform distribution usually can not keep their phase distribution uniform over all periods or even one period, if $n \neq r$. There certainly are some phases corresponding to the observations which nearly or exactly coincide with each other over a certain period. This means that the phase distribution of observations over the period T_i would be nonuniform if Δd and T_i can not be nearly or exactly expressed by the form (41): i.e., a pair of mutual prime numbers n and m_i which satisfy (41) can not be found.

In Chapter 3 we learned the criterion for optimum distribution of observations for determination of the circular error of one wavelength is to keep observations uniform over the whole period. Let us extend the criterion for one wavelength to that for more than one wavelength. The optimum distribution of observations for determining the circular error with a combination of several wavelengths is also a uniform distribution by which all phases of observations are uniformly distributed simultaneously over each period T_i; that is, the interval of observations Δd should satisfy:

(42) $$r \cdot \Delta d = M_1T_1 = M_2T_2 = \ldots = M_kT_k$$

The number of observations r and any number M_i in (42) constitute a pair of mutual prime numbers.

Similarly, the conditions for an optimum distribution over more than one wavelength can be written as follows

$$(43) \qquad \begin{aligned} \sum_{j=1}^{r} \cos 2\Theta_{m,i,j} &= 0 \\ \sum_{j=1}^{r} \sin 2\Theta_{m,i,j} &= 0 \end{aligned} \qquad i = 1, 2, \ldots k$$

It would be impossible to find an interval Δd meeting (43) if there were no common factors between each period T_i, for instance, between the long wavelength $T_1 = 5325$ and $T_2 = 1206$. In practice, a better interval is usually found which is approximately fitted with (42) holding for most of the periods.

4.3.4 Design of the optimum interval for uniformly distributed observations

Based on the formula (42), the optimum interval of observations can be designed. The period T_1 is assumed to be a long wavelength which is nearly equal to the whole range of counter readings, so it is impossible that overlapping phases for all observations occur over the period T_i. Moreover, the period T_7 is assumed to be one mgal which is bound to meet (42). Hence, we do not have to take the periods T_1 and T_7 into account in our design. It is also supposed that there exist some multiple relations between the rest of the periods T_i ($i = 2, 3, \ldots 6$):

$$(44) \qquad \begin{aligned} T_2 &= 17\ T_3 \\ T_3 &= 2\ T_4 \\ T_3 &= 9\ T_5 \\ T_5 &= 2\ T_6 \end{aligned}$$

Using the assumptions, the optimum interval can be determined by the following steps:

1. Specify the number of observations, r.
2. Specify the range of counter readings, R, to be calibrated.
3. Find $m_{2.0} = R/T_2$.
4. Find an integer m_2 which is nearest to $m_{2.0}$ and has no common factor with r except one.
5. $\Delta d_{opt} = m_2 T_2/r$.
6. Specify initial observation d_{off_1} and then compile the table of observations d_{off} by $d_{off_j} = d_{off_{j-1}} + \Delta d_{opt}$ $(j = 1, 2, \ldots r)$

This procedure may be illustrated through the following example. Assume the data:

$r = 44$ observations, $T_2 = 1206$ c.u., $R = 6000$ c.u., $m_{2.0} = 4.98$, $m_2 = 5$, $\Delta d_{opt} = 137.045$ c.u., $d_{off_1} = 100.3$ c.u., $d_{off_j} = d_{off_{j-1}} + 137.045$ c.u. $(j = 2, 3, \ldots 44)$, $W = 180.0$ mgal.

Using the above data, the corresponding standard deviation of the adjusted parameters were obtained by simulative computation: $m_{\delta x} = 0.25 \times 10^{-5}$ mgal/c.u.

	T_1	T_2	T_3	T_4	T_5	T_6	T_7	RMS
$m_{\delta A}$	0.0033	0.00070	0.00031	0.00062*	0.00047*	0.00030	0.00036	0.0013
$m_{\delta \phi}$	0.56	4.14	4.13	8.24*	6.14*	4.02	4.64	5.04
m_y	0.0032	0.00069	0.00031	0.00061	0.00047	0.00030	0.00035	0.0012
m_z	0.0030	0.00069	0.00031	0.00062	0.00046	0.00336	0.00035	0.0012

(*these values of the deviations are larger than the others because

$\sin \frac{\pi W}{T_4 X}$ and $\sin \frac{\pi W}{T_5 X}$ are much less than $\sin \frac{\pi W}{T_i X}$ $(i = 3, 6, 7)$). It would be inconvenient for the gravity meter's operator to reset a counter reading from the observation d_{off_j} to the next observation $d_{off_{j+1}}$ because Δd_{opt} usually is not equal or near to $\Delta d_w = d_{on} - d_{off} = W/x$. Therefore, we had better try to let the value of Δd_{opt} be close to Δd_w.

4.4 Nonuniform Distributions of Observations

4.4.1 The solution for the equations $\partial TR(\Sigma_x)/\partial d_{off} = 0$

In order to find the optimum nonuniform distribution of observations the system of non-linear equations:

$$\begin{aligned} &\frac{\partial\, TR(\Sigma_x)}{\partial\, d_{off_1}} = 0 \\ (45) \qquad &\frac{\partial\, TR(\Sigma_x)}{\partial\, d_{off_2}} = 0 \\ &\vdots \\ &\frac{\partial\, TR(\Sigma_x)}{\partial\, d_{off_r}} = 0 \end{aligned}$$

is solved with an iterative method. From (19) and the expression (20) for the partial derivatives with respect to observation d_{off}, an expansion formula for the left-hand sides of (45) can be obtained. Several sets of roots could be found over the given range of each observation d_{off_j}, so only those roots that make the trace (Σ_x) minimum are taken as the final solutions for (45). The errors of the solutions are believed to be less than 1 c.u.

Since the convergence rate of the iteration for solving the system (45) is quite slow, it would be very expensive to solve a system which includes all wavelengths ($k = 7$) to be calibrated. We therefore only solve those systems (45) which involve the combination of less than 5 wavelengths. The simulative computations should give us some ideas about the distributions. The optimum nonuniform distribution for determination of parameter sets x, Y_i, Z_i, and x, A_i, ϕ_i, are respectively investigated. We are also interested in the phase distribution corresponding to the optimum counter readings with weight off. The phase distributions are illustrated by diagrams similar in nature to those in Figure 2.

4.4.2 The optimum nonuniform distribution for the determination of parameters x, Y_i, and Z_i

Selecting x, Y_i, and Z_i as the parameters for adjustment, the system of equations (45) was solved by the iteration method. Some typical solutions for the equations are compiled in Tables 13a-d. The corresponding optimum phase distributions are illustrated by the Figures 3a-d. In analyzing the results the following facts should be pointed out.

The trace of the variance-covariance matrix of the parameters x, Y_i, and Z_i is reduced through the optimization, and the accuracies of almost all parameters are improved to a certain extent. There are no occasions when the accuracies for some of the parameters are improved while the accuracies of others are getting significantly worse.

If the initial uniform distribution of observations for the iteration is approximately equal to the optimum uniform distribution, then there are practically no differences in the accuracies of the parameters between the optimum uniform distribution and the optimum nonuniform distribution; the

INITIAL d_{off} FOR ITERATION (c.u.)	ROOTS d_{off} (c.u.)	STATISTIC	FOR INITIAL d_{off}	FOR ROOTS
W = 180.0 mgal	r = 15 observations		T_1 = 70.941 c.u.	T_2 = 35.471 c.u.
1000.500	1007.206	$TR(\Sigma_{xyz})$	2.957×10^{-4}	2.847×10^{-4}
	1078.151	$m_{\delta x}$	4.3×10^{-4}	4.3×10^{-4}
1133.833	1208.435	$m_{\delta y1}$	0.0005	0.0005
1267.167	1269.299	$m_{\delta y2}$	0.0011	0.0011
1400.500	1401.240	$m_{o\delta y}$	0.0008	0.0008
	1472.180	$m_{\delta z1}$	0.0006	0.0005
1533.833	1604.709	$m_{\delta z2}$	0.0011	0.0011
1667.167	1737.318	$m_{o\delta z}$	0.0009	0.0008
1800.500	1869.252	$m_{\delta A1}$	0.0006	0.0005
1933.833	1934.493	$m_{\delta A2}$	0.0011	0.0011
	2005.434	$m_{o\delta A}$	0.0009	0.0008
2067.167	2068.059	$m_{\delta\phi 1}$	8.10	8.34
2200.500	2202.250	$m_{\delta\phi 2}$	14.01	14.14
	2273.189	$m_{o\delta\phi}$	11.44	11.61
2333.833	2335.061			
	2406.002			
2467.167	2468.046			
	2538.046			
2600.500	2671.419			
2733.833	2803.587			
2867.167	2936.245			

Table 13a. Some solutions for minimum $TR(\Sigma_x)$ taking x, Y_i, Z_i as the adjusted parameters

INITIAL OBS FOR ITERATION (d_{off})	ROOTS (d_{off})	STATISTIC	FOR INITIAL OBS	FOR ROOTS
	$W = 20.0$ $r = 15$	$T_1 = 7.882$	$T_2 = 3.941$	
2000.650	2002.112	$TR(\Sigma_{xyz})$	0.1575×10^{-4}	0.2849×10^{-5}
2002.650	2004.106	$m_{\delta x}$	3.85×10^{-5}	3.83×10^{-5}
2004.650	2006.174	$m_{\delta y1}$	0.0006	0.0005
2006.650	2007.842	$m_{\delta y2}$	0.0031	0.0011
2008.650	2009.691	$m_{o\delta y}$	0.0022	0.0008
2010.650	2011.762	$m_{\delta z1}$	0.0005	0.0005
2012.650	2013.893	$m_{\delta z2}$	0.0023	0.0011
2014.650	2015.442	$m_{o\delta z}$	0.0017	0.0008
2016.650	2016.650	$m_{\delta A1}$	0.0006	0.0005
2018.650	2018.705	$m_{\delta A2}$	0.0013	0.0011
2020.650	2020.760	$m_{o\delta A}$	0.0010	0.0008
2022.650	2022.676	$m_{\delta\phi 1}$	12.97	13.19
2024.650	2024.650	$m_{\delta\phi 2}$	58.99	17.17
2026.650	2026.650	$m_{o\delta\phi}$	42.71	15.31
2028.650	2028.672			

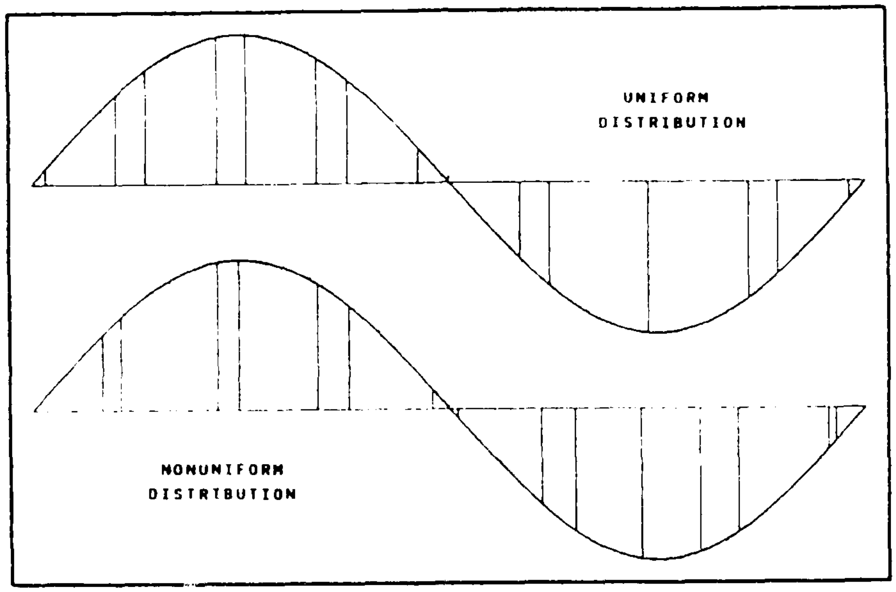
Table 13b. Some solutions for minimum $TR(\Sigma_x)$ taking x, Y_i, Z_i as the adjusted parameters

INITIAL OBS FOR ITERATION (d_{off})	ROOTS (d_{off})	STATISTIC	FOR INITIAL OBS	FOR ROOTS
W = 180.0	r = 21	T_1 = 1206.00	T_2 = 70.941	T_3 = 35.471
1000.700	1048.147	$TR(\Sigma_{xyz})$	0.3566×10^{-2}	0.4075×10^{-5}
1143.557	1267.081	$m_{\delta x}$	40.72×10^{-5}	0.36×10^{-5}
1286.414	1390.340	$m_{\delta y1}$	0.0011	0.0010
1429.271	1560.590	$m_{\delta y2}$	0.0157	0.0005
1572.129	1684.244	$m_{\delta y3}$	0.0177	0.0009
1714.986	1812.344	$m_{o\delta y}$	0.0137	0.0008
1857.843	1911.381	$m_{\delta z1}$	0.0011	0.0010
2000.700	2068.074	$m_{\delta z2}$	0.0486	0.0005
2143.557	2175.790	$m_{\delta z3}$	0.0253	0.0008
2286.414	2350.749	$m_{o\delta z}$	0.0316	0.0008
2429.271	2440.683	$m_{\delta A1}$	0.0011	0.0010
2572.129	2706.558	$m_{\delta A2}$	0.0496	0.0005
2714.986	2748.960	$m_{\delta A3}$	0.0296	0.0009
2857.843	2943.760	$m_{o\delta A}$	0.0334	0.0008
3000.700	3067.046	$m_{\delta\phi 1}$	6.77	6.08
3143.557	3186.691	$m_{\delta\phi 2}$	188.0	7.06
3286.414	3287.745	$m_{\delta\phi 3}$	118.0	11.97
3429.271	3443.746	$m_{o\delta\phi}$	128.2	8.76
3572.129	3690.705			
3714.986	3714.986			
3857.843	3994.513			

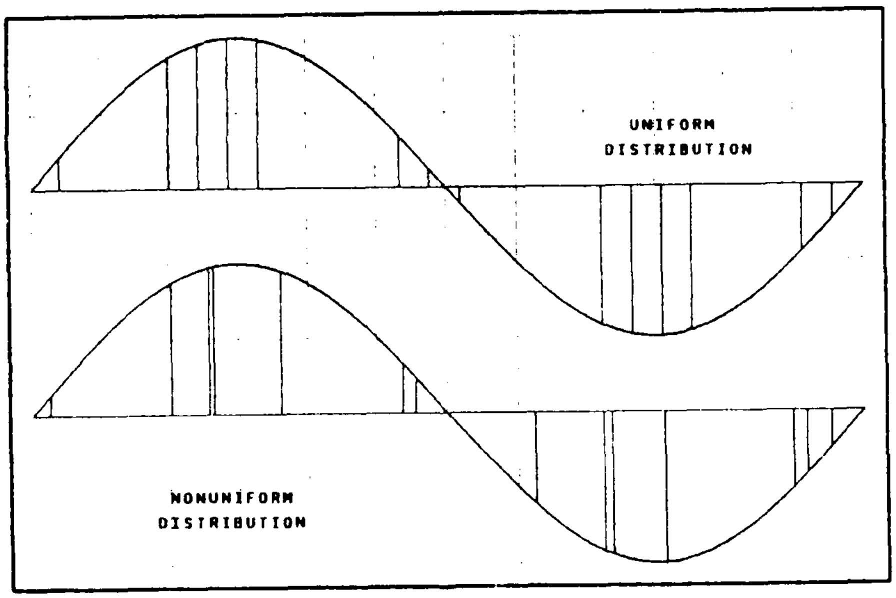
Table 13c. Some solutions for minimum $TR(\Sigma_X)$ taking x, Y_i, Z_i as the adjusted parameters

INITIAL OBS FOR ITERATION (d_{off})	ROOTS (d_{off})	STATISTIC	FOR INITIAL OBS	FOR ROOTS
W = 180.0	r = 21	T_1 = 1206.00	T_2 = 70.941	T_3 = 35.471
100.300	226.884	$TR(\Sigma_{xyz})$	0.1804×10^{-4}	0.4073×10^{-5}
386.014	525.737	$m\delta y$	1.773×10^{-5}	0.360×10^{-5}
671.729	785.606	$m_{\delta y1}$	0.0010	0.0010
957.443	1020.733	$m_{\delta y2}$	0.0016	0.0005
1243.157	1311.965	$m_{\delta y3}$	0.0026	0.0009
1528.871	1596.750	$m_{o\delta y}$	0.0019	0.0008
1814.586	2075.035	$m_{\delta z1}$	0.0010	0.0010
2100.300	2308.249	$m_{\delta z2}$	0.0021	0.0005
2386.014	2490.227	$m_{\delta z3}$	0.0015	0.0009
2671.729	2915.239	$m_{o\delta z}$	0.0016	0.0008
2957.443	3024.489	$m_{\delta A1}$	0.0010	0.0010
3243.157	3310.101	$m_{\delta A2}$	0.0025	0.0005
3528.871	3667.872	$m_{\delta A3}$	0.0024	0.0009
3814.586	3850.079	$m_{o\delta A}$	0.0021	0.0008
4100.300	4166.964	$m_{\delta\phi 1}$	6.24	6.06
4386.014	4420.809	$m_{\delta\phi 2}$	10.53	7.05
4617.729	4739.932	$m_{\delta\phi 3}$	22.93	11.91
4957.443	5026.791	$m_{o\delta\phi}$	15.01	8.72
5243.157	5243.157			
5528.871	5598.793			
5814.586	5888.134			

Table 13d. Some solutions for minimum $TR(\Sigma_x)$ taking x, Y_i, Z_i as the adjusted parameters

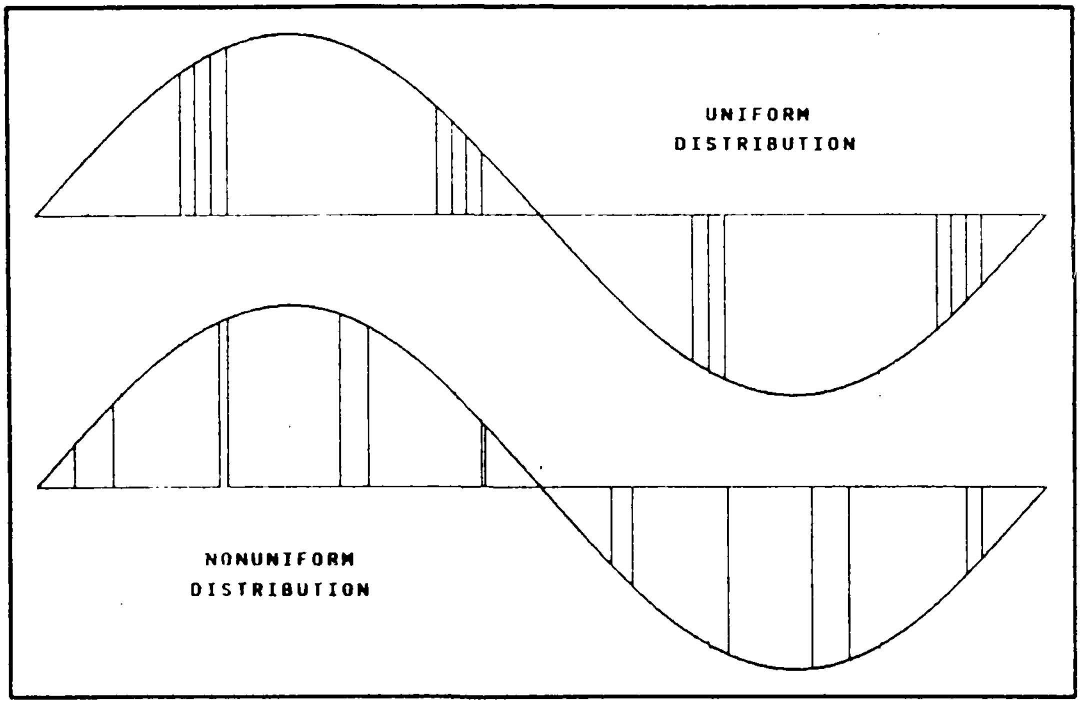


3a. T_1 = 70.941 c.u. r = 15 observations Refer Table 13a

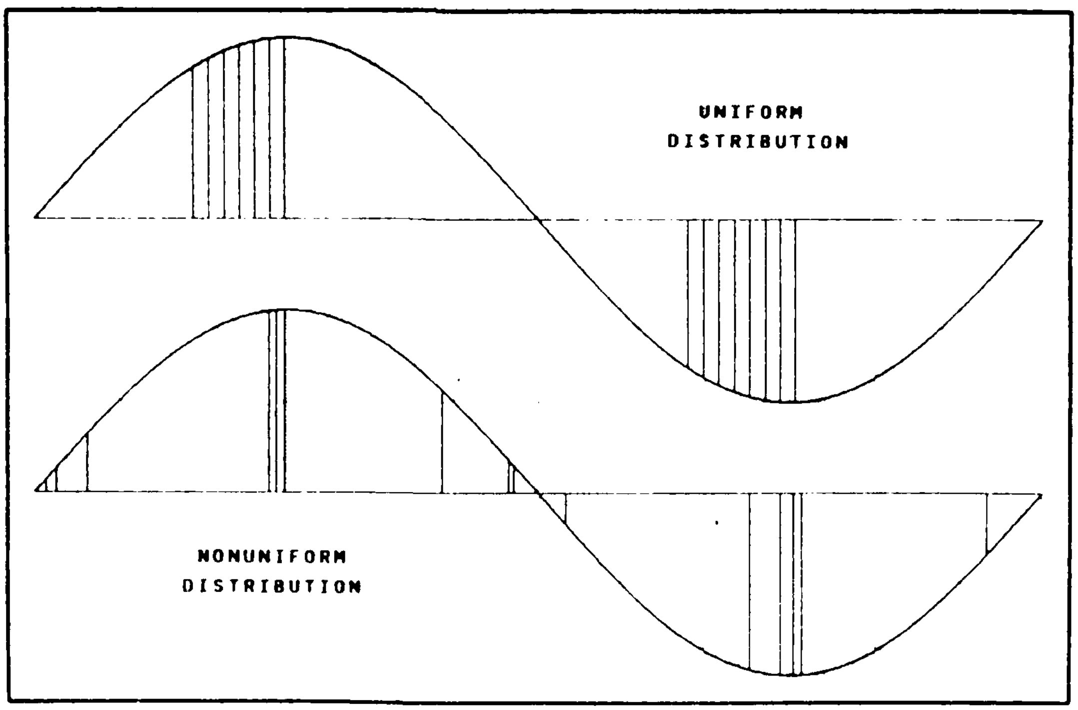


3a. T_2 = 35.471 c.u. r = 15 Refer Table 13a

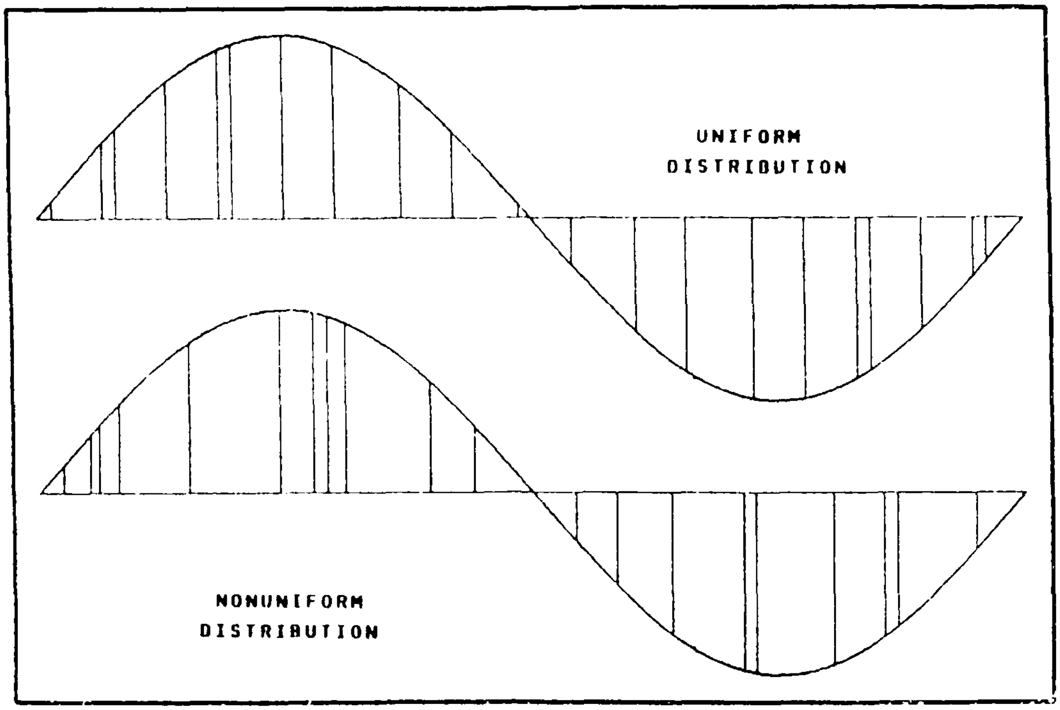
Figure 3. The contrasts between the phase distribution corresponding to uniform distribution of observations (top curve) and optimum nonuniform distribution of observations, (bottom curve) taking x, Y, Z as parameters. (horizontal grid increment is 30°)



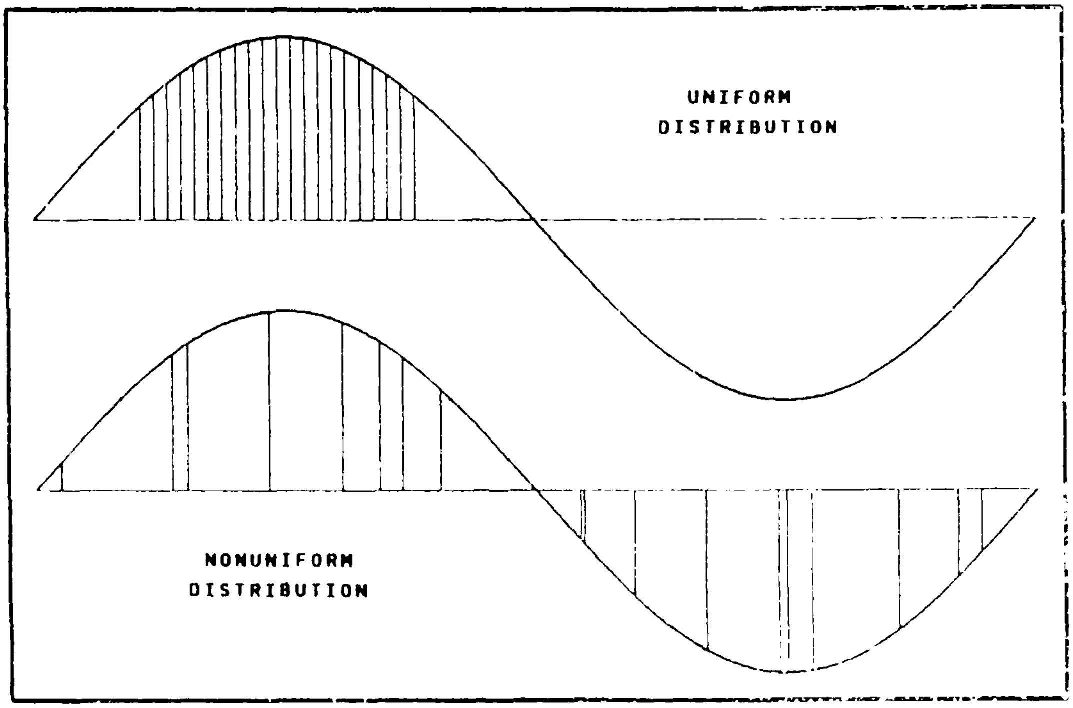
3b. T_1 = 7.882 c.u. r = 15 Refer Table 13b



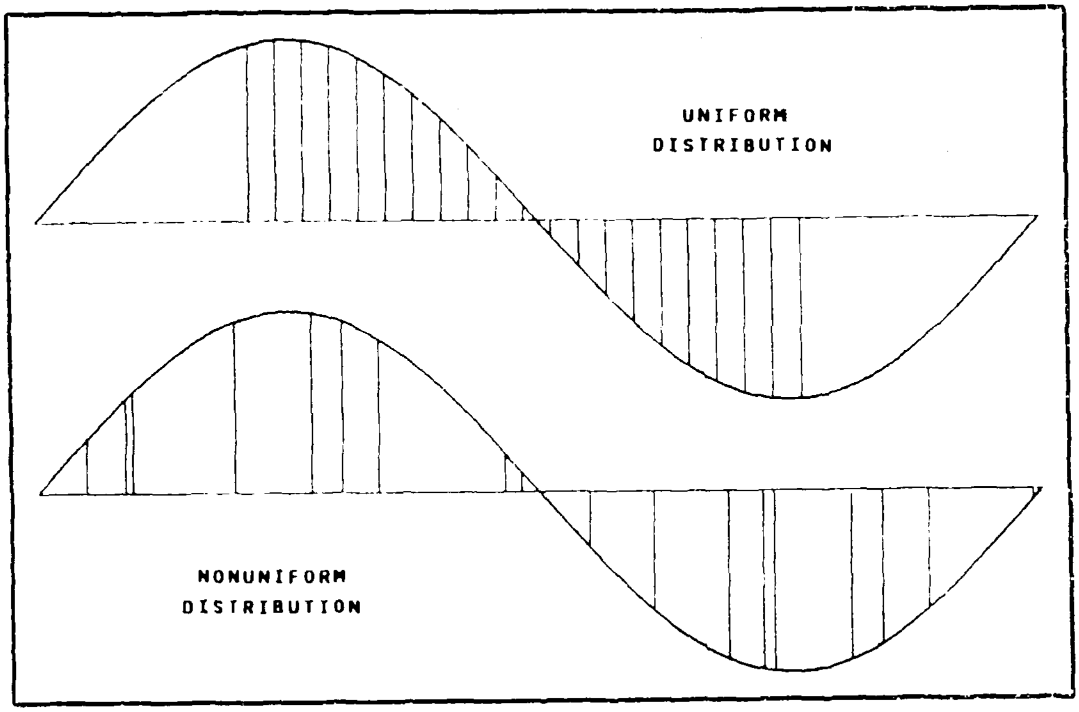
3b. T_2 = 3.491 c.u. r = 15 Refer Table 13b



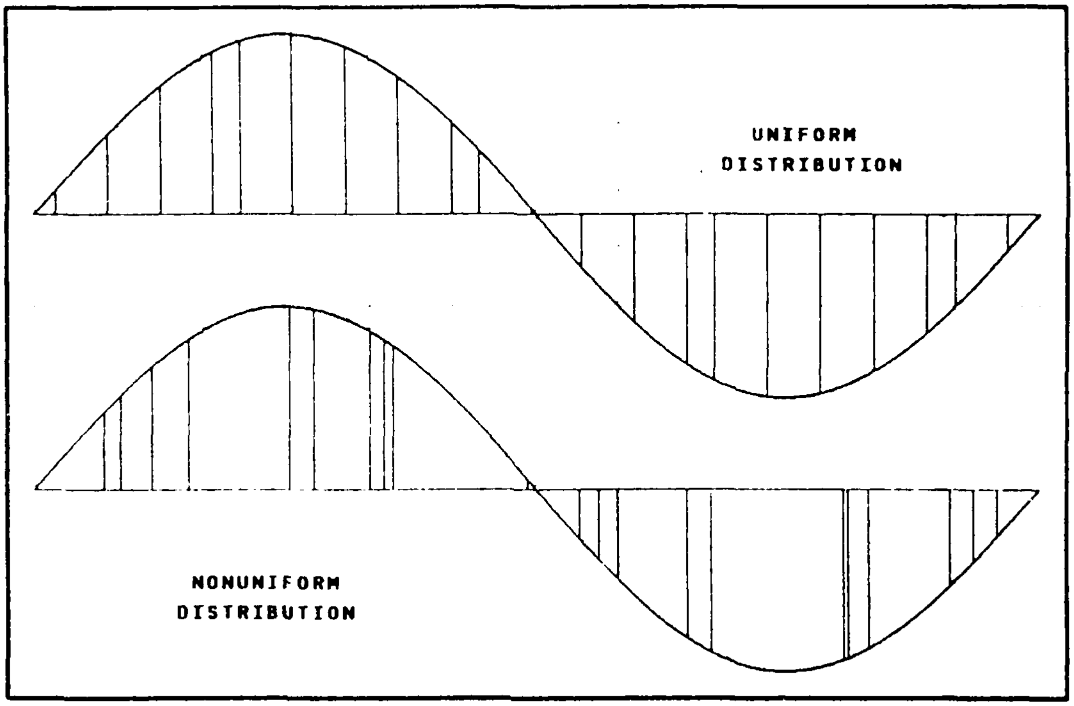
3c. T_1 = 1206 c.u. r = 21 Refer Table 13c



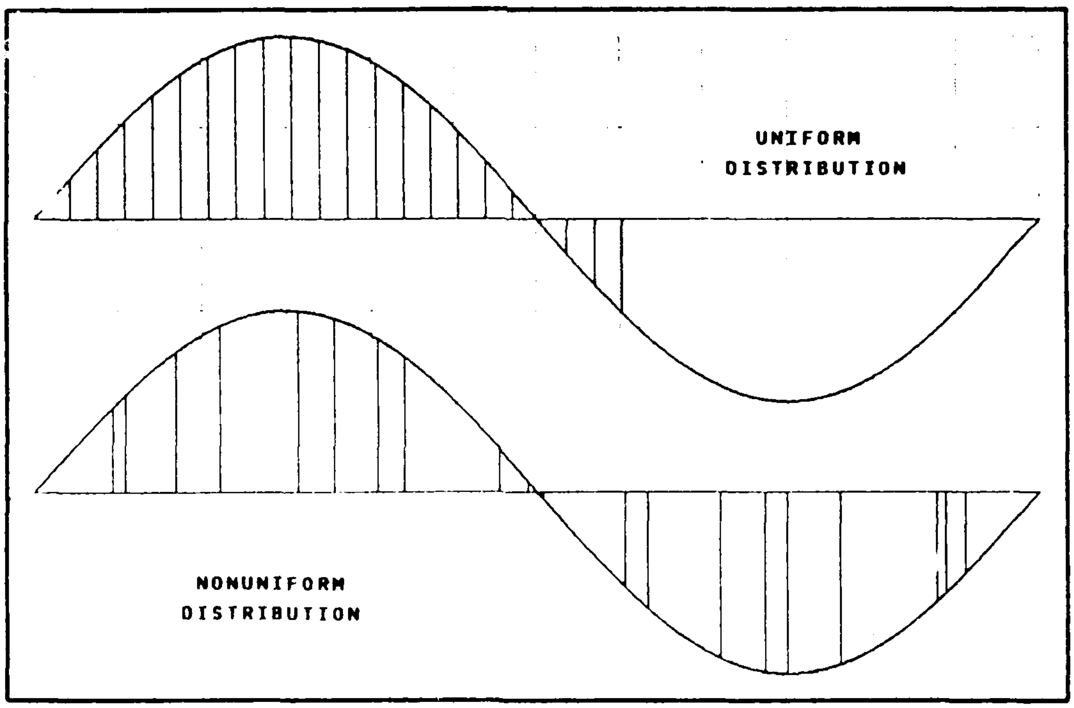
3c. T_2 = 70.941 r_2 = 21 Refer Table 13c



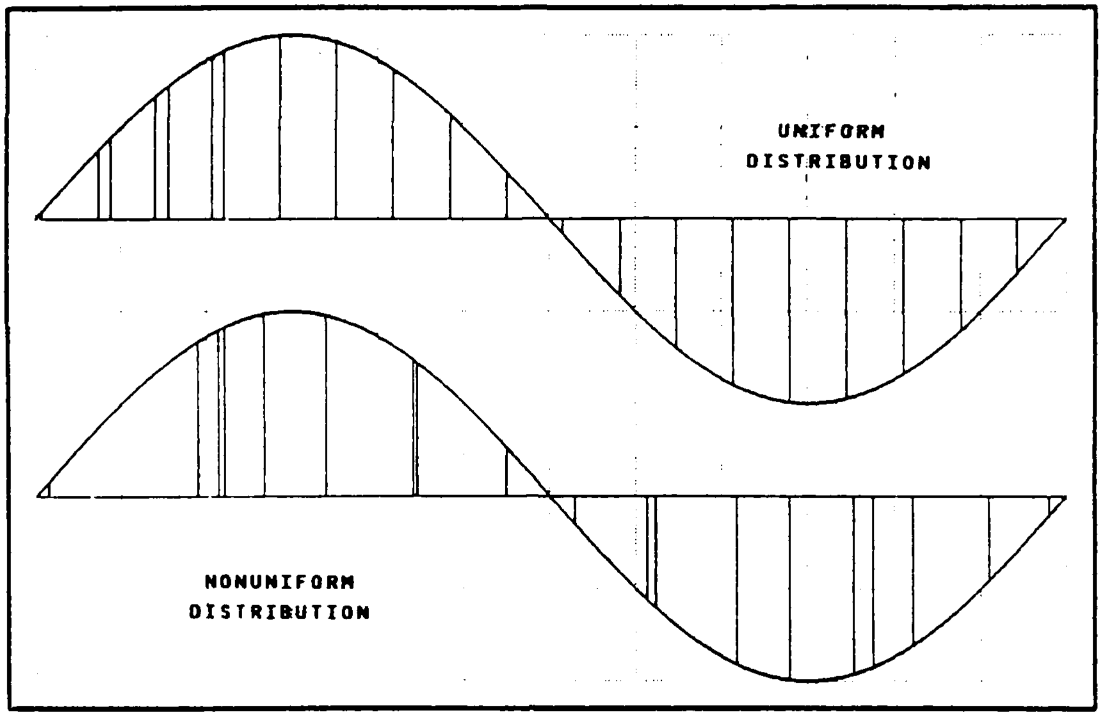
3c. T_3 = 35.471 c.u. r_3 = 21 Refer Table 13c



3d. T_1 = 1206.00 c.u. r = 21 Refer Table 13d



3d. T_2 = 70.941 c.u. r = 21 Refer Table 13d



3d. T_3 = 35.471 c.u. r = 21 Refer Table 13d

change in phase distribution is also not significant (Tables 13a, 13c, and Figures 3a, 3c).

If the initial uniform distribution of observations for the iteration is far from the optimum uniform distribution, namely, the corresponding interval of observations is inadequate, then the accuracies of all parameters will be considerably improved by optimization with the optimum nonuniform distribution of observations. The dense phase distribution located at a few positions over the periods corresponding to the inadequate intervals will nearly uniformly scatter over all periods and change into an optimum nonuniform distribution which is similar to the uniform distribution (of Tables 13b, 13d and Figures 3b, 3d). According to the above facts, we would consider that the accuracies of the parameters x, Y_i, and Z_i obtained from optimization with the uniform distribution of observations can hardly be improved by optimization with the nonuniform distribution.

4.4.3 The character of the optimum nonuniform distribution for determining the parameters x, δA_i and $\delta\phi_i$

Tables 14a-c and Figures 4a-c show the comparison in phase distribution and accuracy between two types of optimum distributions of observations. Several facts are evident. First the trace of the variance-covariance matrix of the parameters x, δA_i and $\delta\phi_i$ formed from the adjustment of nonuniformly distributed observations in each case is markedly less than with the optimum uniform distribution. Secondly, the accuracy of the unknown phase lag correction $\delta\phi_i$ is improved by the adjustment with the optimum nonuniform distribution of observations. The standard deviation $m_{\delta\phi}$ is reduced about 0.5 - 3.0 degrees for these examples. On the other hand a significant reduction f the accuracies of the other parameters δx_i, δA_i, Y_i and Z_i occurred. The

INITIAL OBS FOR ITERATION (d_{off}) (c.u.)	ROOTS (d_{off}) (c.u.)	STATISTIC	FOR INITIAL OBS	FOR ROOTS
W = 180.0 mgal	r = 15 observations		T_1 = 5325.00 c.u.	T_2 = 1206 c.u.
300.300	129.914	$TR(_{xA\phi})$	0.01627	0.01072
700.300	726.907	$m_{\delta x}$	0.43×10^{-5}	0.52×10^{-5}
1100.300	* 900.300	$m_{\delta A1}$	0.0054	0.0054
1500.300	1333.133	$m_{\delta A2}$	0.0012	0.0026
1900.300	1936.407	$m_{o\delta A}$	0.0039	0.0043
2300.300	*2100.300	$m_{\delta\phi 1}$	0.96	0.95
2700.300	2535.219	$m_{\delta\phi 2}$	7.24	5.76
3100.300	3140.650	$m_{o\delta\phi}$	5.16	4.13
3500.300	*3300.300	m_{y1}	0.0052	0.0053
3900.300	3740.352	m_{y2}	0.0012	0.0012
4300.300	4344.585	m_{oy}	0.0038	0.0038
4700.300	4500.300	m_{z1}	0.0051	0.0050
5100.300	4951.614	m_{z2}	0.0012	0.0025
5500.300	5549.644	m_{oz}	0.0037	0.0040
5900.300	*5700.300			

Table 14a. Some solutions for minimum trace (Σ_x) taking x, δA_i and $\delta\phi_i$ as the adjusted parameters.

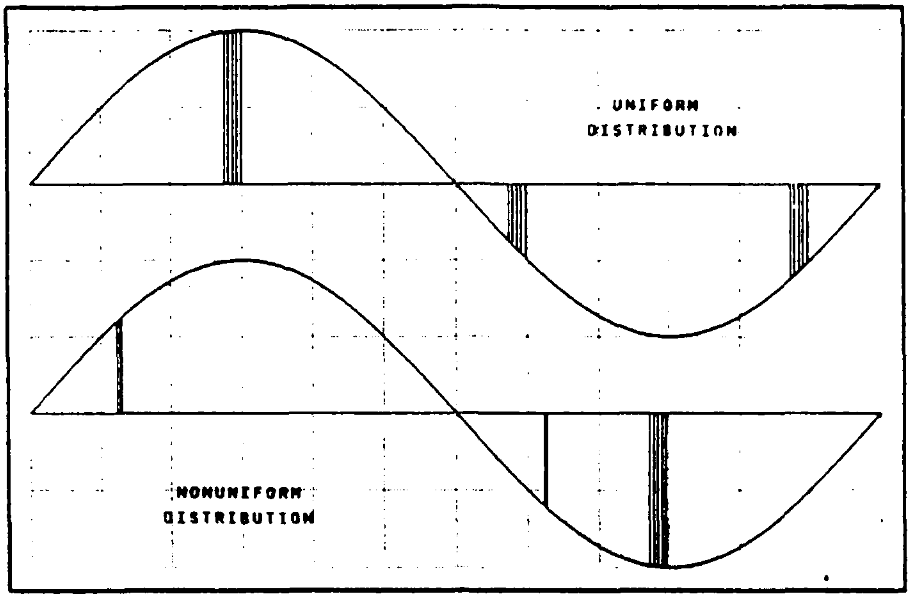
(* not a root of $\frac{\partial TR(\Sigma_x)}{\partial d_{off}} = 0$; this observation is optimum in the given range for finding optimum points.)

INITIAL OBS FOR ITERATION (d_{off})	ROOTS (d_{off})	STATISTIC	FOR INITIAL OBS	FOR ROOTS
	$W = 180.0$ $r = 15$ $T_1 = 70.941$ $T_2 = 35.472$			
2034.033	2027.071	$TR(\Sigma_{xA\phi})$	0.076840	0.05153
2100.700	2118.630	$m_{\delta x}$	0.44×10^{-5}	0.64×10^{-5}
2167.367	2134.658	$r_{\delta A1}$	0.00058	0.00062
2234.033	2225.730	$m_{\delta A2}$	0.00116	0.00351
2300.700	*2267.367	$m_{o\delta A}$	0.00092	0.00252
2367.367	2382.799	$m_{\delta\phi 1}$	8.03	7.53
2434.033	2438.405	$m_{\delta\phi 2}$	13.70	10.60
2500.700	2509.002	$m_{o\delta\phi}$	11.23	9.19
2567.367	2580.209	m_{y1}	0.00051	0.00051
2634.033	2631.732	m_{y2}	0.00106	0.00254
2700.700	2702.695	m_{oy}	0.00083	0.00183
2767.367	2736.943	m_{z1}	0.00058	0.00060
2834.033	2827.802	m_{z2}	0.00114	0.00256
2900.700	2878.777	m_{oz}	0.00090	0.00186
2967.367	2969.362			

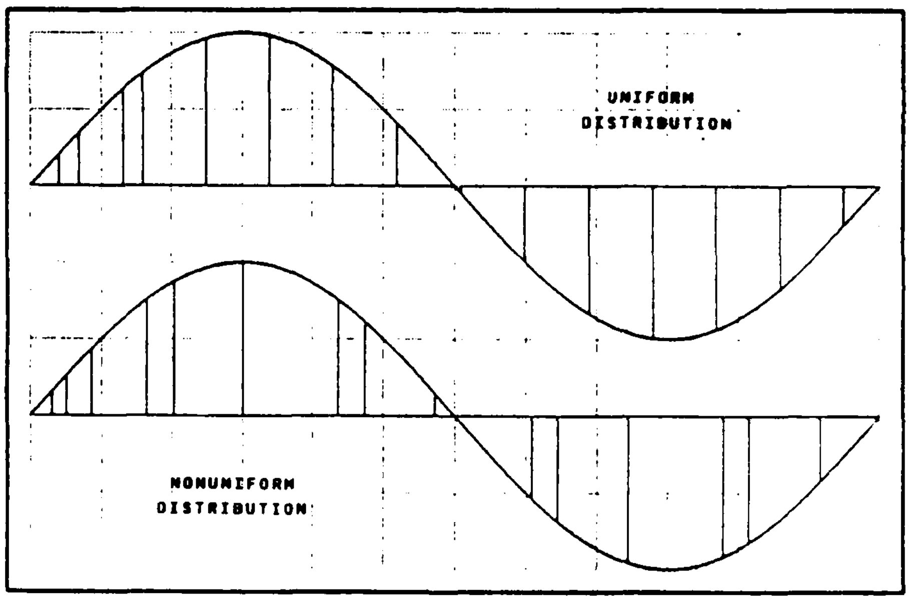
Table 14b. Some solutions for minimum trace (Σ_x) taking x, δA_i and $\delta\phi_i$ as the adjusted parameters.
(*not a root of $\dfrac{\partial TR(\Sigma_x)}{\partial d_{off}} = 0$; this observation is optimum in the given range for finding optimum roots.)

INITIAL OBS FOR ITERATION (d_{off})	ROOTS (d_{off})	STATISTIC	FOR INITIAL OBS	FOR ROOTS
$W = 180.0$	$r = 15$	$T_1 = 7.882$	$T_2 = 3.941$	
2001.650	*2000.650	$TR(\Sigma_{xA\phi})$	0.2070	0.0960
2003.650	2003.785	$m_{\delta x}$	3.85×10^{-5}	5.44×10^{-5}
2005.650	2005.326	$m_{\delta A1}$	0.00053	0.00063
2007.650	2007.509	$m_{\delta A2}$	0.00364	0.00278
2009.650	2009.310	$m_{o\delta A}$	0.00260	0.00201
2011.650	2011.574	$m_{\delta\phi 1}$	13.54	11.75
2013.650	2013.115	$m_{\delta\phi 2}$	22.28	13.30
2015.650	2015.481	$m_{\delta\phi}$	18.44	12.55
2017.650	2017.249	m_{y1}	0.00053	0.00063
2019.650	2019.578	m_{y2}	0.00239	0.00114
2021.650	2021.188	m_{oy}	0.00173	0.00092
2023.650	2023.529	m_{z1}	0.00055	0.00048
2025.650	2025.326	m_{z2}	0.00307	0.00286
2027.650	2027.578	m_{oz}	0.00221	0.00191
2029.650	2029.318			

Table 14c. Some solutions for minimum trace (Σ_x) taking x, δA_i and $\delta\phi_i$ as the adjusted parameters.
(*not a root of $\frac{\partial TR(\Sigma_x)}{\partial d_{off}} = 0$; this observation is optimum in the given range for finding optimum roots.)

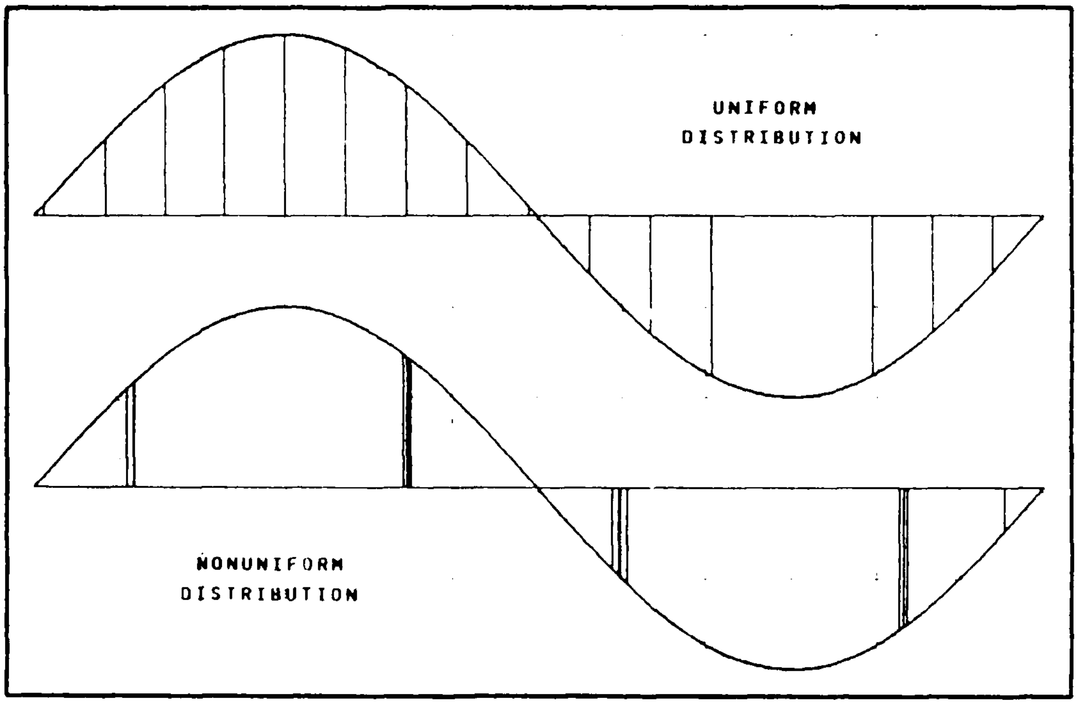


4a. T_1 = 5325.00 c.u. r = 15 observations Refer Table 14a

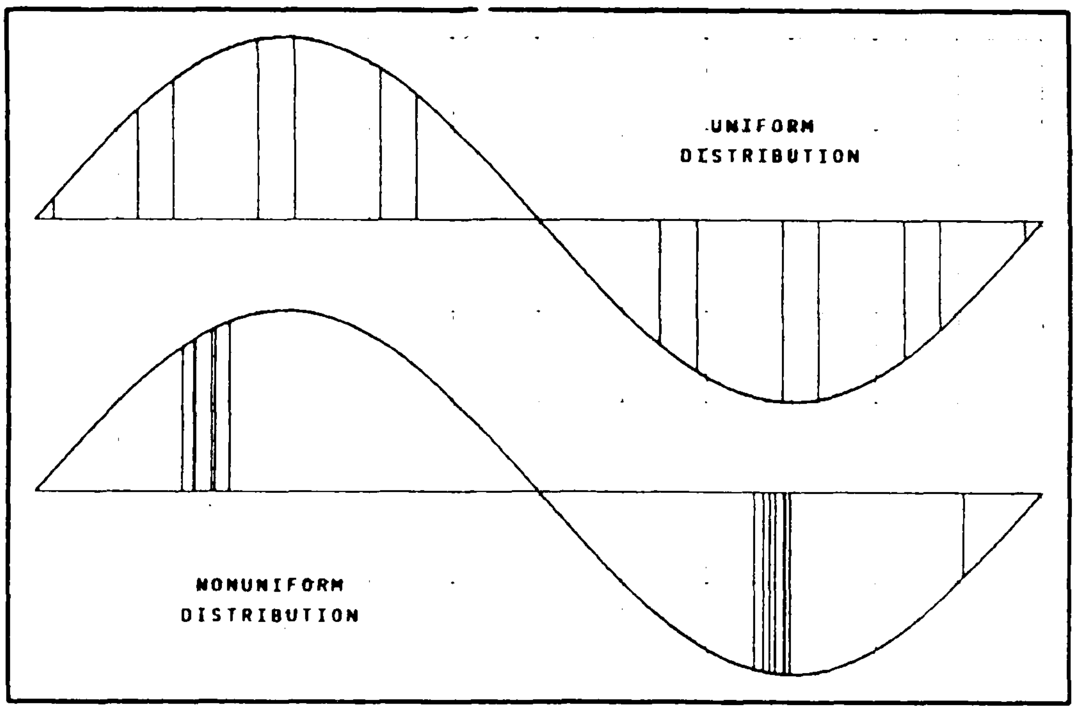


4a. T_2 = 1206.00 c.u. r = 15 Refer Table 14a

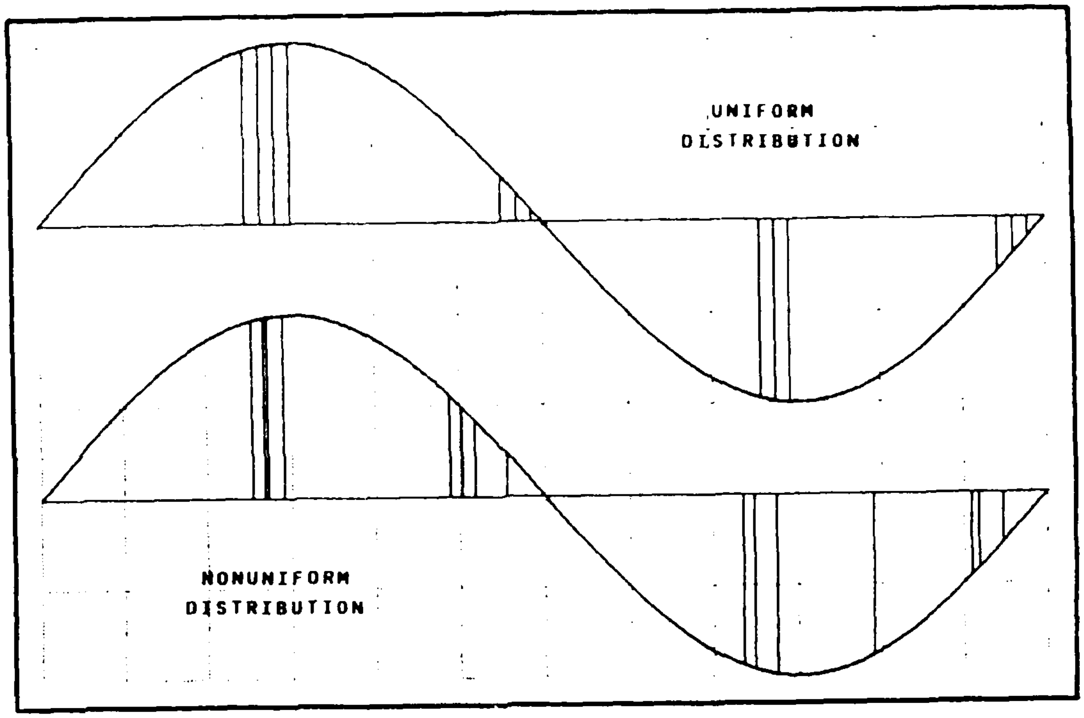
Figure 4. The contrasts between the phase distribution corresponding to uniform distribution of observations and optimum nonuniform distribution of observations, taking x, δA, $\delta \Phi$ as parameters.



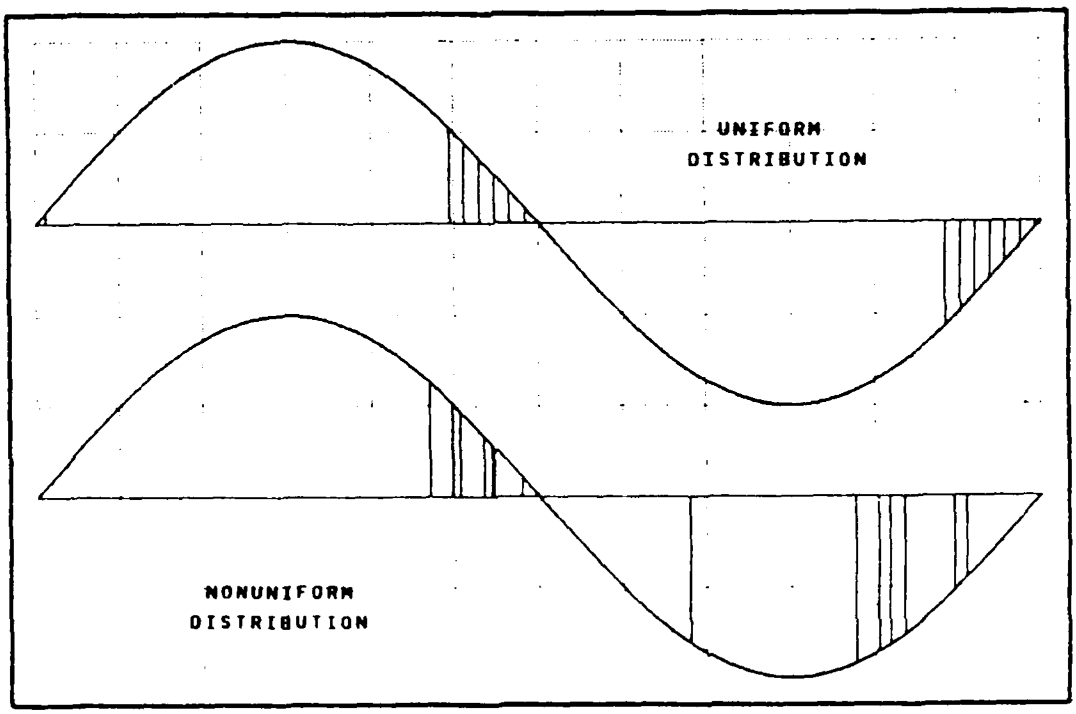
4b. T_1 = 70.941 c.u. r = 15 Refer Table 14b



4b. T_2 = 35.471 r = 15 Refer Table 14b



4c. T_1 = 7.882 c.u.   r = 15   Refer Table 14c



4c. T_2 = 3.941   r = 15   Refer Table 14c

losses in the accuracies of the parameters δA_i, Y_i and Z_i reach about 0.5 - 2.5 μgal, and the linear scale factor δx about 0.2×10^{-5} - 1.6×10^{-5} c.u./mgal. This means that it is advantageous only for the determination of the phase lag ϕ_i to solve the system (45). However, it is just in the sense of comparing optimally distributed uniform and nonuniform observations.

In order to realize this situation mentioned above, let us analyze the character of the optimum nonuniform distribution of observations. In Figs. 4a-c, there is a common character to the optimum nonuniform distribution. That is, most of the observations are concentrated near $\Theta_m = 0°$ and 180° over each period with the exception of the long wavelengths, which resembled the phase distribution of the uniform distribution.

The phase distribution for wavelengths T_3 and T_5 are almost the same. The mean positions of some dense distributions represented by average phase angle Θ_m, are approximately equal to 45°, 150°, 225° and 330°. For wavelengths T_4 and T_6, $\Theta_m = 15°$ and 165°. Regarding wavelength T_2, the corresponding phase distribution also tends toward 0° and 180°, but not strongly.

In a typical observation equation, the coefficient for $\delta\phi_i$ has a factor A_i which is usually much less than 1 mgal. It is obvious that the accuracy of determination of $\delta\phi_i$ would be improved if the value of $\cos(\frac{2\pi}{T_i} d_m - \phi_i)$ were increased: i.e. if $\Theta_m = \frac{2\pi}{T_i} d_m - \phi_i = 0°$ or 180°. Otherwise, the coefficient for $\delta\phi_i$ would be small and the accuracy of the adjusted $\delta\phi_i$ would correspondingly decrease. This implies that the variance of $\delta\phi$ is more sensitive to the phase Θ_m than the variance δA. Concerning wavelength T_1, the corresponding phase distribution is nearly uniform. There is no significant difference in the accuracy of δA and $\delta\phi$, and especially $\delta\phi$, between the optimum nonuniform

and uniform distributions. As discussed previously, a uniform distribution of observations is preferable for determination of scale factor x and amplitude A_i.

It was pointed out in section 3.2 that the condition $\sin 2\theta_{mj} = 0$ (j = 1, 2, ... r) must be imposed for the simultaneous minimization of $TR(\delta A_i)$ and $TR(\delta\phi_i)$. However, this type of conditional distribution could not be satisfied for all wavelengths in the periodic error function for either a nonuniform or uniform distribution. This would be one of the reasons why the simultaneous minimization of $TR(\delta A_i)$ and $TR(\delta\phi_i)$ could not generally be reached in the optimization procedure, especially in the ones with a nonuniform distribution of observations. For the uniform distribution, however, a simultaneous minimization of $TR(\delta A_i)$ and $TR(\delta\phi_i)$ is nearly realized, since both traces decrease at comparable rates in the optimization procedure.

In addition, it is very important that the maximum effect of $m_{\delta A}$ on converting counter readings to their milligal equivalent is itself $m_{\delta A}$. However, the error of ϕ, $m_{\delta\phi}$, causes the maximum error of the milligal equivalent to be A $m_{\delta\phi}$. For example, if $m_{\delta\phi} = 3° = 0.05$ (rad) and A = 10 μgal, then A $m_{\delta\phi} = 0.5$ μgal. As is mentioned previously, a 3 degree decrease of $m_{\delta\phi}$ would result in a 3 μgal increase of $m_{\delta A}$ in the optimization procedure with the nonuniform distribution. In view of this point, it seems impossible, in practice, to obtain an optimum distribution for highly accurate adjusted parameters, if only the optimization procedure under a nonuniform distribution of observations is employed.

4.5 Requisite Number of Observations

Usually, a complete procedure of calibration expends much time, for instance, 2-3 weeks, and therefore involves appreciable expense. We should consider the important question, "how many observations are needed for determining the circular error with a reasonable accuracy?" With respect to one wavelength, an answer has been given to this question in Section 3.3. Concerning the combination of more than one wavelength, it is difficult to answer the question exactly and absolutely. However, an approximate estimate of the requisite number of observations can be formed by some simulative computations.

First the accuracy criterion has to be given for the estimation of the number of observations used in the simulative computation. The accuracy of determination of amplitude for the calibration of an LCR "G" gravity meter has been in the range of about 0.4 μgal to 6.5 μgal, and 2.0 μgal in mean value. The accuracy of determination of phase lag in the "G" gravity meter has ranged from 3° to 29°, 14° in the mean value (Kanngieser and Torge, 1981). As is known, the minimum division of the dial in a "G" gravity meter is roughly equivalent to 10 μgal. The maximum error of the readings can be considered to be 0.005 c.u. It would be preferable that the RMS $m_{\delta A_i}$ should be estimated by the RMS of A_i involved in the periodic screw error function. According to the empirical data from the calibration for LCR "G" gravity meters (Kanngieser and Torge, 1981), the range of the amplitudes have been found which are as follows:

< 5 μgal for the 1 c.u. period

< 2 μgal for the 3.9 c.u. period

< 9 μgal for the 7.9 c.u. period

< 8 μgal for the 35 c.u. period

$<$ 20 μgal for the 71 c.u. period

$<$ 20* μgal for the 1206 c.u. period

(* found using the fewest measurements). The root mean square root of A_i is 10.7 μgal. If $A_i\, m_{\delta\phi_i} < 1.0$ μgal, the limiting RMS of $\delta\phi_i$, $m_{\delta\phi_i}$ are as follows:

$<$ 11° for the 1 c.u. period

$<$ 29° for the 3.9 c.u. period

$<$ 6° for the 7.9 c.u. period

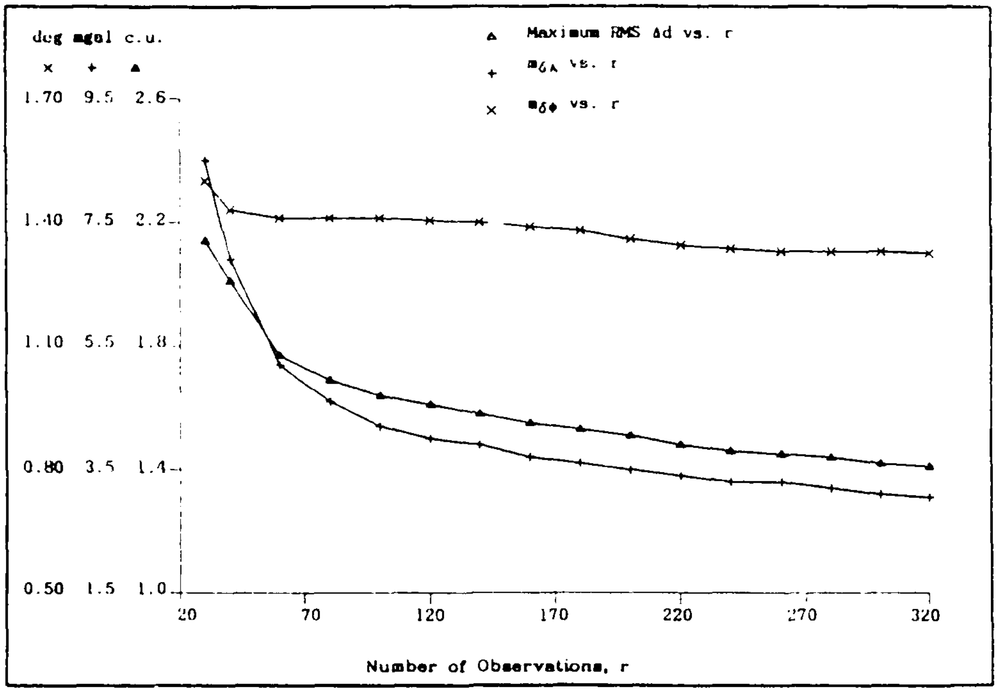
$<$ 7° for the 35 c.u. period

$<$ 3° for the 71 c.u. period

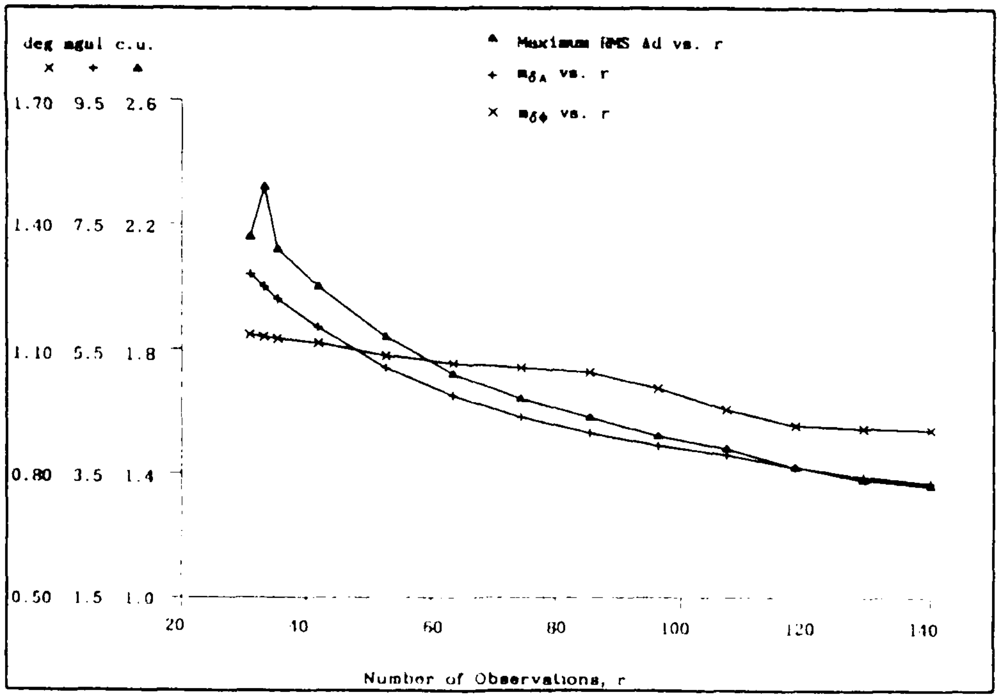
$<$ 3° for the 1206 c.u. period

The root mean square of $m_{\delta\phi_i}$ is required to be less than 13°. From the amplitudes given for simulative computation in this report, the limiting $m_{\delta\phi}$ is 15°, i.e., $m_{\delta\phi} < 15°$.

Figures 5a-h show some relationships between the number of observations and the RMS values of the amplitudes and phase lags, as well as the maximum RMS in the RMS of the adjusted quantity $\Delta d = d_{on} - d_{off}$. The curves in Figure 5a are drawn with the data from the simulative computation under current calibration procedures and apparatus. 30 observations using the 180 mgal weight are made at first, and then we increase sequentially the observations using the 20 mgal weight at intervals of 2 observations up to more than 300. It is found that a decrease of $m_{\delta A}$ from 1.43 to 1.32 μgal occurs along with the increase of the number of observations from 40 up to 324. In other words, an increase of 284 observations brings about an improvement in the accuracy of adjusted parameters δA_i only by 0.11 μgal, and $m_{\delta A_i}$ has remained over 1.3 μgal. The increase of observations using only a small weight, such as 20

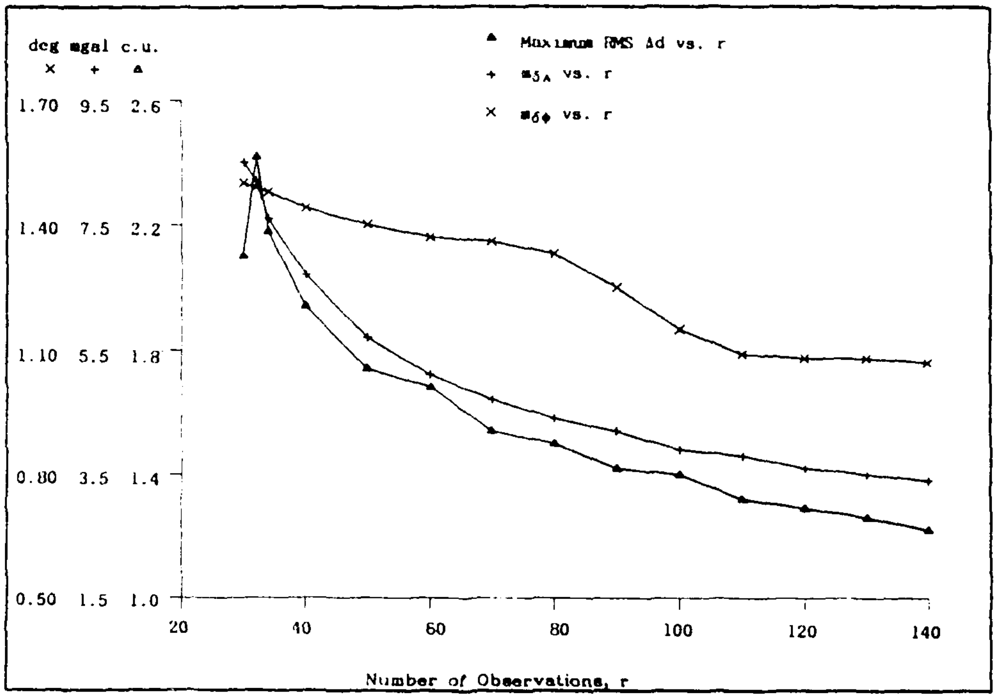


5a. W_1 = 180.0 mgal, W_2 = 20.0 mgal, Δd_1 = 171.650
Δd_2 = 19.072, r_1 = 30, r_2 = r - 30

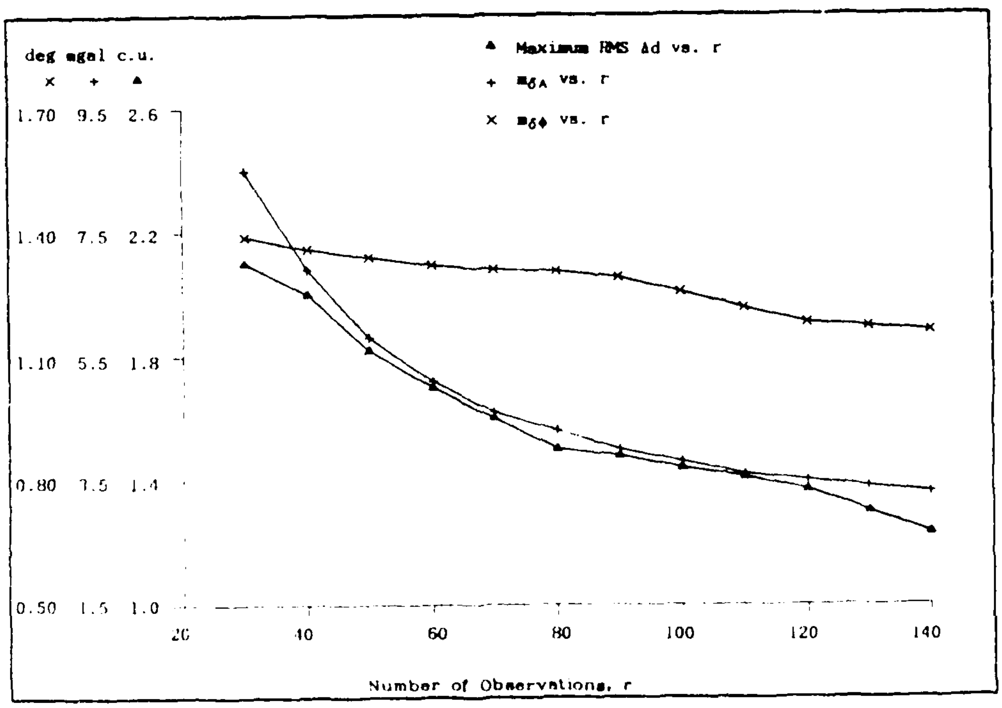


5b. W_1 = 242.672 mgal, W_2 = 88.540 mgal, Δd_1 = 153.301
Δd_2 = 61.569, r_1 = 30, r_2 = r - 30

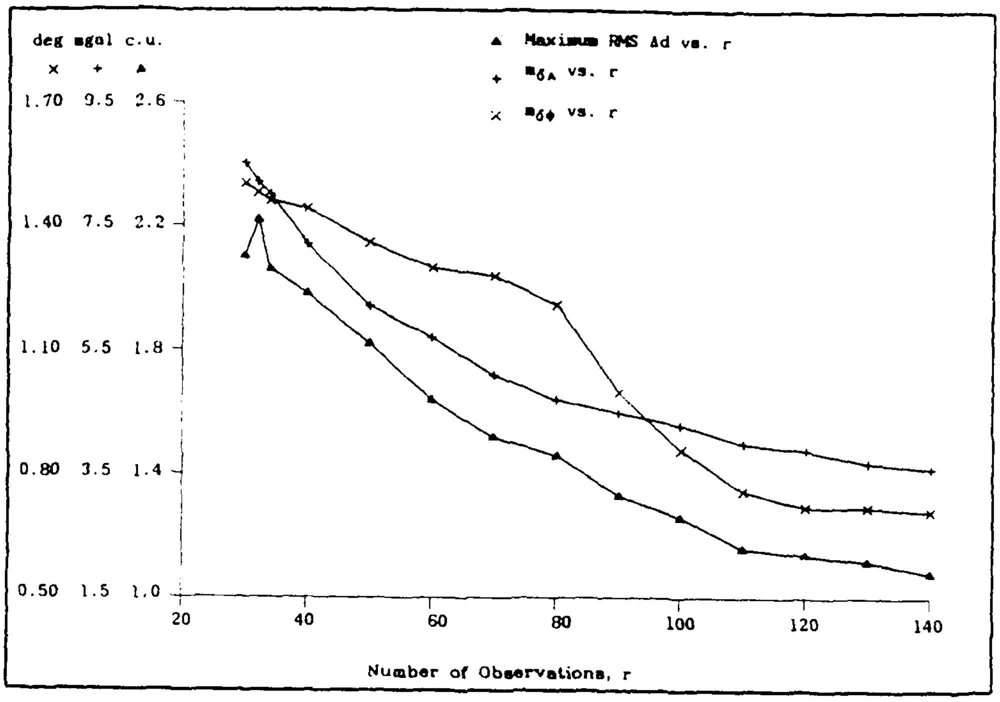
Figure 5. The relationship between the standard deviation of adjusted parameters and the number of observations.



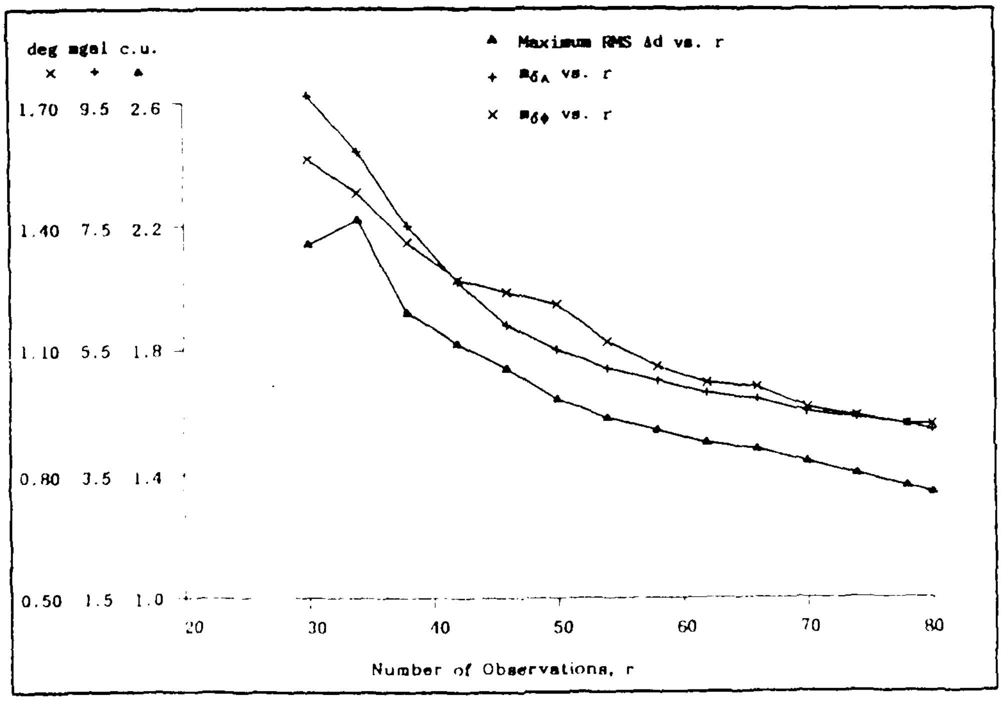
5c. W_1 = 180.0 mgal, W_2 = 88.54 mgal, Δd_1 = 171.65
Δd_2 = 61.569, r_1 = 30, r_2 = r − 30



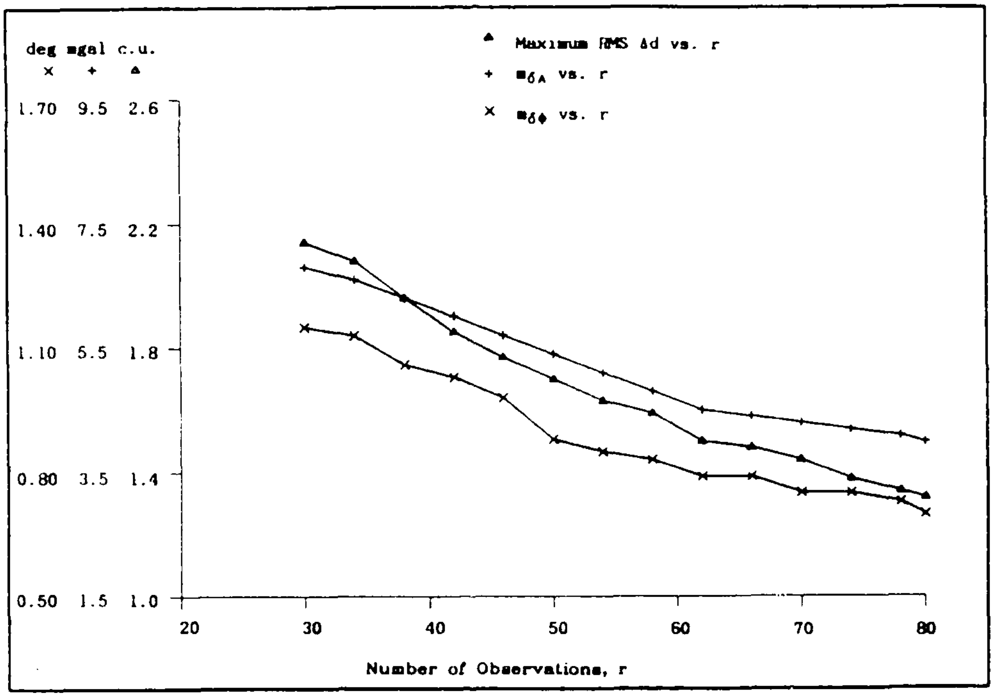
5d. W_1 = 200.0 mgal, W_2 = 55.01 mgal, Δd_1 = 190.722
Δd_2 = 52.458, r_1 = 30, r_2 = r − 30



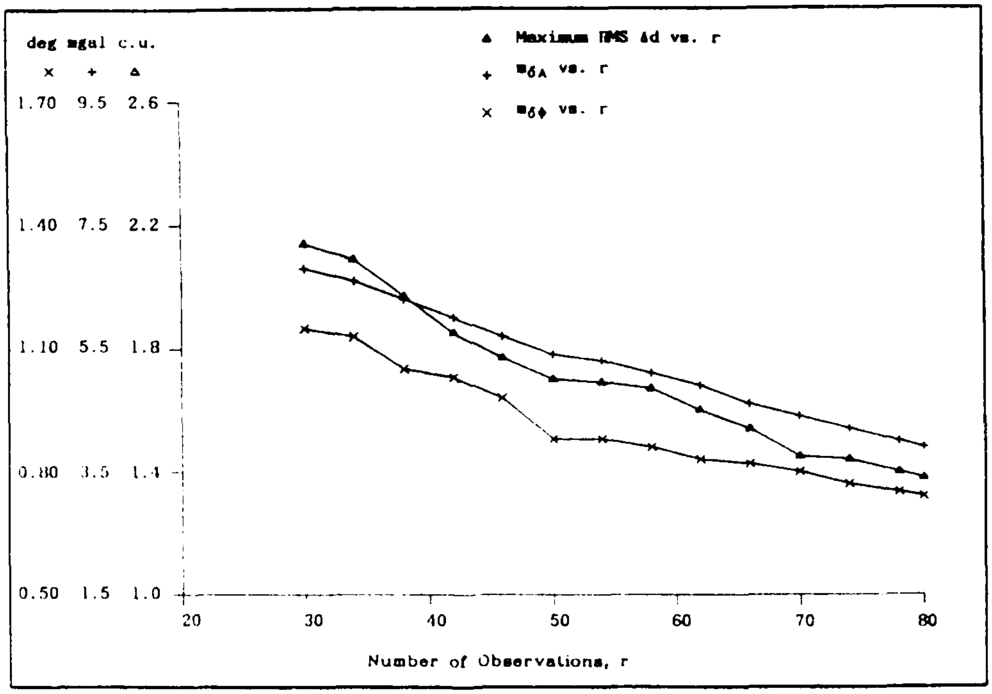
5e. W_1 = **180.0 mgal,** W_2 = **180.0 mgal,** Δd_1 = **171.65**
Δd_2 = **58.854,** r_1 = **30,** $r_2 = r - 30$



5f. W_1 = **178.962 mgal,** W_2 = **178.962 mgal,** Δd_1 = **170.66**
Δd_2 = **170.66,** r_1 = **30,** $r_2 = r - 30$



5g. W_1 = 242.672 mgal, Δd_1 = 171.650



5h. W_1 = 242.672 mgal, W_2 = 180.0 mgal, Δd_1 = 231.415
Δd_2 = 171.65, r_1 = 30 - 50, r_2 = r - 50

mgal, could not lead to a significant improvement in the accuracy of parameter δA_i. However, the improvement in the accuracy appears remarkable as the observations using a large weight, for instance, 180 mgal, are increased.

Let us have a look at Figs. 5a-h, the figures indicate:

$r > 60$, $m_{\delta A} < 1.3$ μgal
$r > 90$, $m_{\delta A} < 1.0$ μgal for a weight of 180.0 mgal

$r > 40$, $m_{\delta A} < 1.3$ μgal
$r > 66$, $m_{\delta A} < 1.0$ μgal for a weight of 178.962 mgal

$r > 30$, $m_{\delta A} < 1.2$ μgal
$r > 45$, $m_{\delta A} < 1.0$ μgal for a weight of 242.672 mgal

If $m_{\delta A} < 1.3$ μgal is required, about 40 to 60 observations, for which a large weight should be adopted, are needed. If $m_{\delta A} < 1.0$ μgal, about 70 to 90 observations are needed. In this case, the maximum RMS of the adjusted milligal equivalent to the counter readings difference $d_{on} - d_{off}$ would be less than about 1.8 μgal. If a larger weight, for instance, 242.672 mgal were used instead of the 180.0 mgal weight, the number of required observations could be less than 45 to meet the mentioned accuracy. From Figures 5 we note that about 80 to 90 observations would be enough when $m_{\delta A} < 1.3$ μgal is specified. This requires 30 extra observations using a weight larger than 180 mgal, and 50 to 60 extra ones using a smaller weight of about 50 to 80 mgal. According to the simulative computations in Sections 4.2 and 4.3, the requisite number of observations would be less than the above estimated number of observations required, provided the weight and the distribution of observations are selected and designed by the criteria of optimization.

Concerning the requisite observations for determining adjusted parameter phase lag ϕ with the reasonable accuracy of say $m_{\delta\phi} < 13°$, it is obvious, from the figures that the accuracy would be easily obtained provided about 30 to 40 observations are made with a proper interval for uniform distribution and using an appropriately large weight in the calibration. We emphasize that the intervals and weights have to be carefully selected based on those criteria that were discussed in Sections 4.2 and 4.3. In addition, the $m_{\delta\phi} - r$ curves in the figures show that the values of $m_{\delta\phi}$ are basically inversely proportional to the number of observations.

Finally, we point out that the results of this section just come from simulative computations with a specific model for the periodic screw error function, where it is assumed that the standard deviation of an observation is equal to 0.002 c.u. If there would be some change in these conditions for practical calibration, the results might have some small differences.

5. CONCLUSION

The sizes of the weights used in calibration have significant effects on the accuracies of all adjusted parameters of the periodic screw error function, since the standard errors of the parameters are sensitive to the quantity $\sin(\pi W/T_i x)$. The optimum weights for detecting the periodic errors should have the sizes of nearly $(n_i + \frac{1}{2})\ T_i x$; namely, the value of $\sin(\pi\ W/T_i x)$ should be approximately one. The larger the value is, the better the corresponding weight. In the case of calibrating the periodic screw error function with one wavelength, the optimum size of the weight is equal to $(n + \frac{1}{2})$ Tx, no matter what type of parameters are selected in the

adjustment. For several wavelengths, it is impossible to find only one weight having the special size for fitting each period T_i involved. As a general rule, the optimum weights could be found by solving the equation $\partial TR(\Sigma_x)/\partial W = 0$ for minimizing $TR(\Sigma_x)$. The study in this report shows that the solutions for the equation are approximately equivalent to the solutions for the equation $\partial(\sum_{i=1}^{k} \sin \frac{\pi W}{T_i})/\partial W_o = 0$. For two wavelengths fulfilling $T_1 = 2T_2$, the roots of both equations are expressed by $W_{opt} = (n + 0.32)\ T_1\ x$. For more than two wavelengths, in order to obtain higher and homogeneous accuracy, it is preferable that one weight W_1 and another compensative weight W_2 are adopted so that the value $|\sin(\pi\ W_1/T_i x)| + |\sin(\pi\ W_2/T_i x)|$ is greater than 1.6 (see Tables 10 and 11), except for long wavelengths ($i = 1, 2$).

Since the selection for the weights is related to the linear scale factor x and there are some differences in scale factors between different gravity meters, a meter should have its own weights as an accessory of the meter. In addition, a large weight can be also used for detecting a very short periodic screw error provided the corresponding value $|\sin(\pi W/Tx)|$ is great enough. A large weight is better than a small weight for improving the accuracy of linear scale factor x and amplitude A. If a non-linear term of the scale factor, such as quadratic term, can be neglected, it would be appropriate to employ one or two large weights for the calibration.

The optimum distribution of observations for calibrating the periodic screw error with one wavelength is a uniform distribution. If $Y = A\cos\phi$ and $Z = A\sin\phi$ are taken as parameters in the adjustment, the optimum uniform distribution has to satisfy the condition expressed by the equations $\sum_{j=1}^{r} \cos 2\theta_{om_j} = \sum_{j=1}^{r} \sin 2\theta_{om_j} = 0\ \left(\theta_{om_j} = \frac{\pi}{T}(d_{off_j} + d_{on_j})\right)$ over the range of one period. In this case, the highest accuracy for determining the

parameter's amplitude A and phase lag ϕ would not be attained, especially when A is very small. However, the difference in accuracy between the two sets of unknown parameters is not significant.

For determining the parameters A and ϕ, the optimum distribution of observations is required to satisfy the stronger conditions $\sin 2\theta_{mj} = 0$ ($\theta_{m_j} = \frac{\pi}{T}(d_{off_j} + d_{on_j}) - \phi$) for minimizing both of $TR(\Sigma_{\delta A})$ and $TR(\Sigma_{\delta\phi})$. The condition means that the distribution of observation is defined by $d_{m_j} = (d_{off_j} + d_{on_j})/2 = (n/4 + \phi/2\pi)\ T$.

With regard to the optimum distribution of observations for detecting the periodic errors with more than one wavelength, two types of distributions, uniform and nonuniform, are respectively discussed. For the uniform distribution, the interval of observations should fit in with all periods appearing in the periodic error function, especially with some middle and short wavelengths. Otherwise the number of effective observations would be much less than the practical number of observations. As a matter a fact, it is the phase distribution over the range of periods involved and corresponding to observations that acts in determining the standard errors of the adjusted parameters and not the distribution of observations over the range of counter readings. The uniform distribution of observations could result in nonuniform phase distribution. The number of separate phases, which is referred to as the number of effective observations, is always less than the number of practical observations, or at most equal to them. The number of the phases n distributed over a period T_i can be found by the expression $n\ \Delta d = M_i \cdot T_i$ (n and M_i are two mutual prime numbers). If the number n is much less than the number of observations r, then the corresponding interval Δd is considered to be inadequate for the distribution. When the number n is just equal to the

number r for all periods occurring in the periodic error function, the corresponding interval Δd is optimum, so is the corresponding distribution. The optimum Δd_{opt} can be determined if some multiple relation exists between the periods (see Section 4.3.4). For LCR "G" gravity meter having the old gear box, relations between the periods of 1206, 1206/17, 1206/34, 1206/153, 1206/306 and 1 are present. All observations are always effective for the very long period being equal nearly to the total range of 7000 mgal. The optimum intervals for uniform distribution can also be found by solving the equation $\partial TR(\Sigma_x)/\partial \Delta d = 0$. The solutions for the equations are basically consistent with those intervals that are found from the expression $r\ \Delta d = M_i\ T_i$.

With regard to the nonuniform distribution of observations, the simulative computations for optimization indicate that the optimum nonuniform distribution of observations does not have more advantages for an improvement in accuracy of adjusted parameters than the optimum uniform distribution. If it is the trace of the covariance matrix of the parameters x, Y and Z that is minimized in optimization, there is no significant difference in accuracies of the parameters between the optimum nonuniform distribution of observations and the optimum uniform distribution.

Minimization of the trace of covariance matrix of the parameters x, δA and $\delta\phi$ in optimization is not preferable for improving accuracy of all parameters entering into the adjustment. Although the accuracy of the parameter $\delta\phi$ can be improved significantly in the optimization, the accuracies of the parameters x and δA would become worse, and even below the requirement of the accuracy for the calibration. The reason is that the trace $(\Sigma_{\delta\phi})$ is more sensitive to the phase distribution of observations than trace $(\Sigma_{\delta A})$ in the process of optimization by solving to the equation $\partial TR(\Sigma_x)/\partial d_{off} = 0$. The

phase angles corresponding to the observations would be concentrated on some special phase positions over a period, such as 0°, 180°, etc. The dense phase distribution results in reducing the accuracy of the parameters x and δA which basically represent the accuracy of the conversion from counter readings into milligal equivalents.

This leads to the conclusion that so called optimum nonuniform distribution of observations would not be preferable for the calibration to the optimum uniform ones, in the sense of optimization criterion discussed previously.

How many observations are needed for the calibration of periodic screw errors with reasonable accuracy? For detecting the error with one wavelength, the study indicates that the $TR(\Sigma_{\delta A})$ and $TR(\Sigma_{\delta\phi})$ are both inversely proportional to the number of observations r made with optimum distribution and using optimum calibrating weights. If the RMS error of δA, $m_{\delta A}$, is required to be less than 0.001 mgal, and $\sigma = 0.002$ c.u. is assumed, then the number of observations made in the calibration has to be more than 5 in theory. In general, letting $m_{\delta A} = A\, m_{\delta\phi}$, then the requisite number of observations r for calibrating the periodic error with one wavelength can be approximately estimated by the formula $r > (x\sigma/(\sin \Delta\theta\varepsilon))^2$, where ε is a specified accuracy for adjusted parameter A and $\Delta\theta = \pi W/T$.

Concerning the problem about necessary observations for calibrating the errors with more than one wavelength, the estimate is made by the simulative studies. Based on the error of a counter reading of an LCR "G" gravity meter and the empirical data of standard errors of adjusted parameters documented by a great deal of practical calibrations for LCR "G" gravity meter (Kanngieser and Torge 1981, and Becker, 1981), it would be appropriate that the error $m_{\delta A_i}$

or $A_i m_{\delta\phi_i}$ is required to be less than $5/\sqrt{2k}$ μgal, where k is the number of the periods involved in the periodic errors. Assuming k = 7, then $m_{\delta A_i}$ and $A_i m_{\delta\phi_i}$ should be less than 1.3 μgal or 1.0 μgal, and $m_{\delta\phi_i}$ less than 13°, which are considered as an accuracy criterion for the estimate.

The simulative studies indicate that when a large weight, such as 180.0 mgal, is used in the calibration, the RMS errors of all adjusted parameters x, δA and δϕ rapidly decrease with the increase of observations. But the increase of observations only using a small weight, e.g. 20.0 mgal, can hardly result in an improvement in the accuracy of parameters δA and x. For instance, an increase of 284 observations from 40 to 324 including 30 observations using weight 180.0 mgal, brings about an improvement in RMS error of δA only by 0.14 μgal and the RMS error of δA has remained more than 1.3 μgal, and RMS error of linear scale factor x keeps near a constant 0.03×10^{-5} mgal/c.u. ($\sigma = 0.02$ mgal).

If a non-linear term of scale factor is negligible, those observations that are made with a large weight are considered for estimating the requisite number of observations. By the simulative studies, about 40 to 60 observations using nearly 180 mgal weight are needed for keeping the RMS error of δA less than 1.3 μgal, and about 70 to 90 observations for keeping it less than 1.0 μgal. If a weight larger than 180 mgal, such as 242.672 mgal is adopted, about 35 observations are enough for having $m_{\delta A}$ less than 1.3 μgal, and about 45 observations for having $m_{\delta A}$ less than 1.0 μgal.

The requisite observations could be about 5 to 10 less than the above estimated number of observations required when the weight and the intervals for uniform distribution of observations are strictly selected by the criteria for optimization presented in the report.

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APPENDIX A

Derivation of the formulae for the partial derivatives of Δd_o, ΔC and ΔS with respect to d_{off}, Δd and W

The function for converting a counter reading d to its milligal equivalent D is:

$$D = xd + \Sigma A\cos(\lambda d - \phi) = f(d)$$

The inverse of the function f(d) is $d = F(D)$

$$f'(d) = \frac{\partial D}{\partial d} = x - \Sigma\lambda A\sin(\lambda d - \phi)$$

Let $m = f'(d)$, m_1 (or m_{off}) $= f'(d_{off})$, m_2 (or m_{on}) $= f'(d_{on})$

$$F'(D) = \frac{\partial d}{\partial D} = \frac{1}{f'(d)} = \frac{1}{m}$$

$$d_{on} = F(D_{on}) = F(f(d_{off}) + W) = F(D_{off} + W)$$

$$\Delta d_o = d_{on} - d_{off}$$

$$\Delta d_j = d_{off_{j+1}} - d_{off_j}$$

The above expressions are the basic formulae for the derivation below

$$\frac{\partial \Delta d_o}{\partial d_{off}} = \frac{\partial(d_{on} - d_{off})}{\partial d_{off}} = \frac{\partial d_{on}}{\partial d_{off}} - 1$$

$$= \frac{\partial F(D_{on})}{\partial d_{off}} - 1 = \frac{\partial F(D_{on})}{\partial D_{on}} \frac{\partial D_{on}}{\partial d_{off}} - 1$$

$$= F'(D_{on})f'(d_{off}) - 1 = \frac{f'(d_{off})}{f'(d_{on})} - 1 = \frac{m_{off}}{m_{on}} - 1$$

$$\frac{\partial \Delta d_{o_j}}{\partial \Delta d_j} = \frac{\partial \Delta d_{o_j}}{\partial d_{off_j}} \cdot \frac{\partial d_{off_j}}{\partial \Delta d_j} = \left(\frac{f'(d_{off_j})}{f'(d_{on_j})} - 1\right) \frac{\partial d_{off_j}}{\partial \Delta d_j} = \left(\frac{m_{off_j}}{m_{on_j}} - 1\right) \frac{\partial d_{off_j}}{\partial \Delta d_j}$$

$$d_{off_j} = d_{off_1} + (j - 1)\, \Delta d_j = d_{off_1} + (j - 1)\, \Delta d$$

(assume $\Delta d_1 = \Delta d_2 = \ldots = \Delta d_{r-1}$)

$$\frac{\partial d_{off_j}}{\partial \Delta d_j} = j - 1$$

$$\frac{\partial \Delta d_{o_j}}{\partial \Delta d_j} = \left(\frac{m_{off_j}}{m_{on_j}} - 1\right) (j - 1)$$

$$\frac{\partial \Delta d_{o_j}}{\partial W} = \frac{\partial (d_{on_j} - d_{off_j})}{\partial W} = \frac{\partial d_{on}}{\partial W} = \frac{\partial d_{on_j}}{\partial D_{on_j}} \frac{\partial D_{on_j}}{\partial W}$$

$$= F'(D_{on_j})\ \frac{\partial (D_{off} + W)_j}{\partial W} = \frac{1}{f'(d_{on_j})} = \frac{1}{m_{on_j}}$$

(note: $\frac{\partial (D_{off} + W)}{\partial W} = 1$)

$$\frac{\partial \Delta C_{ij}}{\partial d_{off_j}} = \frac{\partial}{\partial d_{off_j}} \left(\cos(\lambda_i d_{on_j} - \phi_i) - \cos(\lambda_i d_{off_j} - \phi_i)\right)$$

$$= -\lambda_i \sin(\lambda_i d_{on_j} - \phi_i) \frac{\partial d_{on_j}}{\partial d_{off_j}} + \lambda_i \sin(\lambda_i d_{off_j} - \phi_i)$$

$$\frac{\partial d_{on_j}}{\partial d_{off_j}} = \frac{\partial d_{on_j}}{\partial D_{on_j}} \frac{\partial D_{on_j}}{\partial d_{off_j}} = F'(D_{on_j}) \frac{\partial (D_{off_j} + W)}{\partial d_{off_j}}$$

$$= F'(D_{on_j}) f'(d_{off_j}) = \frac{m_{off_j}}{m_{on_j}}$$

$$\frac{\partial \Delta C_{ij}}{\partial d_{off_j}} = -\lambda_i \sin(\lambda_i d_{on_j} - \phi_i) \frac{m_{off_j}}{m_{on_j}} + \lambda_i \sin(\lambda_i d_{off_j} - \phi_i)$$

$$\frac{\partial \Delta C_{ij}}{\partial \Delta d_j} = \frac{\partial \Delta C_{ij}}{\partial d_{off_j}} \frac{\partial d_{off_j}}{\partial \Delta d_j} = \frac{\partial \Delta C_{ij}}{\partial d_{off_j}} (j - 1)$$

$$\frac{\partial \Delta C_{ij}}{\partial W} = \frac{\partial}{\partial W} (\cos(\lambda_i d_{on_j} - \phi_i) - \cos(\lambda_i d_{off_j} - \phi_i))$$

$$= \frac{\partial}{\partial W} (\cos(\lambda_i d_{on_j} - \phi_i)) = -\lambda_i \sin(\lambda_i d_{on_j} - \phi_i) \frac{\partial d_{on_j}}{\partial W}$$

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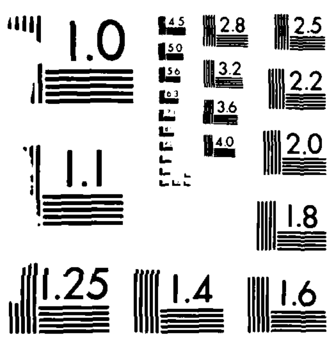

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$$\frac{\partial d_{on_j}}{\partial W} = \frac{\partial d_{on_j}}{\partial D_{on}} \frac{\partial D_{on}}{\partial W} = \frac{1}{m_{on_j}} \frac{\partial (D_{off} + W)}{\partial W} = \frac{1}{m_{on_j}}$$

$$\frac{\partial \Delta C_{ij}}{\partial W} = - \frac{\lambda_i \sin (\lambda_i d_{on_j} - \phi_i)}{m_{on_j}}$$

similarly, $\frac{\partial \Delta S_{ij}}{\partial d_{off_j}}$, $\frac{\partial \Delta S_{ij}}{\partial \Delta d_j}$ and $\frac{\partial \Delta S_{ij}}{\partial W}$

can be derived.

APPENDIX B

Derivation of the formulae for the Trace (Σ_x) formed by the adjustment for determining the parameters of a periodic screw error function with one wavelength

The formulae for the Trace (Σ_x) is:

$$TR(\Sigma_x) = \frac{2\sigma_o^2 \, x^2 \left[\sum_{j=1}^{r} \Delta S_{o_j}^2 + \sum_{j=1}^{r} \Delta C_{o_j}^2 \right]}{\sum_{j=1}^{r} \Delta S_{o_j}^2 \sum_{j=1}^{r} \Delta C_{o_j}^2 - \left(\sum_{j=1}^{r} \Delta \Sigma S_{o_j} \, \Delta C_{o_j} \right)^2}$$

$$= \frac{2\sigma_o^2 \, x^2 \, r}{\sin^2 \Delta\Theta \left[r^2 - \left(\sum_{j=1}^{r} \cos 2\Theta_{om_j} \right)^2 - \left(\sum_{j=1}^{r} \sin 2\Theta_{om_j} \right)^2 \right]}$$

expand the function f(d) (see appendix A)

$$f(d) = D = x\,d + A \cos\left(\frac{2\pi}{T} d - \phi\right)$$

$$= x\,d + A\cos(\phi)\cos\left(\frac{2\pi}{T} d\right) + A \sin(\phi)\sin\left(\frac{2\pi}{T} d\right)$$

$$= x\,d + Y \cos\left(\frac{2\pi}{T} d\right) + Z \sin\left(\frac{2\pi}{T} d\right)$$

where D = milligal equivalent

d = counter reading (observable)

$Y = A\cos\phi$

adjusted parameters

$Z = A\sin\phi$

The condition equation is:

$$x\,\Delta d + \left(\cos\left(\frac{2\pi}{T}\,d_{on}\right) - \cos\left(\frac{2\pi}{T}\,d_{off}\right)\right) Y +$$

$$\left(\sin\left(\frac{2\pi}{T}\,d_{on}\right) - \sin\left(\frac{2\pi}{T}\,d_{off}\right)\right) Z - W_o = 0$$

or $$\Delta C\,Y + \Delta S\,Z + W = 0 \qquad (W = x\,\Delta d - W_o)$$

$$\Delta C_j = \left[\cos\left(\frac{2\pi}{T}\,d_{on_j}\right) - \cos\left(\frac{2\pi}{T}\,d_{off_j}\right)\right]$$

$$\Delta S_j = \left[\sin\left(\frac{2\pi}{T}\,d_{on_j}\right) - \sin\left(\frac{2\pi}{T}\,d_{off_j}\right)\right]$$

$$A = \begin{bmatrix} \Delta C_1 & \Delta S_1 \\ \Delta C_2 & \Delta S_2 \\ \vdots & \\ \Delta C_r & \Delta S_r \end{bmatrix} ; \quad B = x \; {}_r\!\left[\begin{array}{cccc|cccc} 1 & & & & -1 & & & \\ & 1 & \cdot & 0 & & -1 & \cdot & 0 \\ & 0 & \ddots & & & 0 & \ddots & \\ & & & 1 & & & & -1 \end{array}\right]_{2r}$$

$$BB^T = 2x^2\,{}_rI_r \;; \quad (BB^T)^{-1} = \frac{1}{2x^2}\,{}_rI_r \;; \quad ({}_rI_r = \text{unit matrix})$$

$$N = A^T(BB^T)^{-1}A = \frac{1}{2x^2}\begin{bmatrix} \Sigma\Delta C^2 & \Sigma(\Delta C\Delta S) \\ \Sigma(\Delta C\Delta S) & \Sigma\Delta S^2 \end{bmatrix}$$

$$N^{-1} = \frac{2x^2}{\Sigma\Delta C^2\;\Sigma\Delta S^2 - (\Sigma\Delta S\;\Delta C)^2}\begin{bmatrix} \Sigma\Delta S^2 & -\Sigma(\Delta C\Delta S) \\ -\Sigma(\Delta C\Delta S) & \Sigma\Delta C^2 \end{bmatrix}$$

$$TR(\Sigma_x) = \sigma_o^2\;TR(N^{-1}) = \frac{2\sigma_o\;x^2\;[\Sigma\Delta S^2 + \Sigma\Delta C^2]}{\Sigma\Delta S^2\;\Sigma\Delta C^2 - (\Sigma\Delta S\;\Sigma\Delta C)^2}$$

Let $\Theta_1 = \frac{2\pi}{T} d_{off}$; $\Theta_2 = \frac{2\pi}{T} d_{on}$; $\Theta_m = \frac{1}{2}(\Theta_1 + \Theta_2)$

$\Delta\Theta = \frac{1}{2}(\Theta_2 - \Theta_1) = \text{constant}$

(Because $d_{on} - d_{off} = W/x$)

$$\Sigma \Delta S^2 = \Sigma(\sin\Theta_2 - \sin\Theta_1)^2 = \Sigma(2\cos\Theta_m \sin\Delta\Theta)^2 = 4\sin^2\Delta\Theta\ \Sigma\cos^2\Theta_m$$

$$\Sigma \Delta C^2 = \Sigma(\cos\Theta_2 - \cos\Theta_1)^2 = \Sigma(-2\sin\Theta_m \sin\Delta\Theta)^2 = 4\sin^2\Delta\Theta\ \Sigma\sin^2\Theta_m$$

$$\sum_1^r \Delta S^2 + \sum_1^r \Delta C^2 = 4\sin^2\Delta\Theta \sum_1^r (\sin^2\Theta_m + \cos^2\Theta_m) = 4\sin^2\Delta\Theta\ r$$

$$\Sigma\ \Delta S^2\ \Sigma\Delta C^2 = (4\sin^2\Delta\Theta\ \Sigma\cos^2\Theta_m)\ (4\sin^2\Delta\Theta\ \Sigma\sin^2\Theta_m)$$

$$= 16\sin^4\Delta\Theta\ (\Sigma\cos^2\Theta_m\ \Sigma\sin^2\Theta_m)$$

$$= 16\sin^4\Delta\Theta\ \{\Sigma[\tfrac{1}{2}(1 + \cos2\Theta_m)]\ \Sigma[\tfrac{1}{2}(1 - \cos2\Theta_m)]\}$$

$$= 4\sin^4\Delta\Theta \sum_1^r (1 + \cos2\Theta_m) \sum_1^r (1 - \cos2\Theta_m)$$

$$= 4\sin^4\Delta\Theta\ (r + \Sigma\cos2\Theta_m)\ (r - \Sigma\cos2\Theta_m)$$

$$= 4\sin^4\Delta\Theta\ [r^2 - (\Sigma\cos2\Theta_m)^2]$$

$$(\Sigma\Delta S\ \Delta C)^2 = (\Sigma\ 4\ \sin^2\Delta\Theta\ \sin\Theta_m\ \cos\Theta_m)^2$$

$$= 16\ \sin^4\Delta\Theta\ \left(\sum_1^r \frac{1}{2}\ \sin 2\Theta_m\right)^2$$

$$= 4\ \sin^4\Delta\Theta\ \left(\sum_1^r \sin 2\Theta_m\right)^2$$

Hence: $\Sigma\Delta S^2\ \Sigma\Delta C^2 - (\Sigma\Delta S\ \Delta C)^2 = 4\ \sin^4\Delta\Theta\ [r^2 - (\Sigma\ \cos 2\Theta_m)^2 - (\Sigma\ \sin 2\Theta_m)^2]$

$$\frac{\Sigma\Delta S^2 + \Sigma\Delta C^2}{\Sigma\Delta S^2\ \Sigma\Delta C^2 - (\Sigma\Delta S\ \Delta C)^2} = \frac{4\ \sin^2\Delta\Theta\ r}{4\ \sin^4\Delta\Theta\ [r^2 - (\Sigma\ \cos 2\Theta_m)^2 - (\Sigma\ \sin 2\Theta_m)^2]}$$

$$= \frac{r}{\sin^2\Delta\Theta\ [r^2 - (\Sigma\ \cos 2\Theta_m)^2 - (\Sigma\ \sin 2\Theta_m)^2]}$$

(note: Σ denotes $\sum_{j=1}^{r}$)

when taking δA and $\delta\phi$ as adjusted parameters, similarly, the formula for $TR(\Sigma_x)$,

$$Tr(\Sigma_x) = \frac{\sigma_o^2\ x^2[(1 + A^2)r + (A^2 - 1)\ \sum_{j=1}^{r} \cos 2\Theta_{m_j}\ ]}{A^2\sin^2\Delta\Theta\ \{r^2 - [(\sum_{j=1}^{r}\ \cos 2\Theta_{m_j})^2 + (\sum_{j=1}^{r}\ \sin 2\Theta_{m_j})^2]\}}$$

can be also derived by the above procedure.

APPENDIX C

Observations d_{on} and d_{off} (in counter units) for the four numerical examples reported in Section 4.3.1

	Example 1		Example 2	
	d_{off}	d_{on}	d_{off}	d_{on}
1.	499.9969	671.6531	499.9969	671.6531
2.	679.9990	851.6742	671.6490	843.3302
3.	859.9995	1031.6756	843.2995	1014.9977
4.	1039.9965	1211.7094	1014.9465	1186.6433
5.	1220.0014	1391.6961	1186.6014	1358.3138
6.	1400.0014	1571.6943	1358.2514	1529.9426
7.	1579.9985	1751.6857	1529.8985	1701.5983
8.	1759.9991	1931.6956	1701.5491	1873.2659
9.	1939.9988	2111.6868	1873.1988	2044.8900
10.	2120.0022	2291.7102	2044.8522	2216.5620
11.	2300.0013	2471.6957	2216.5013	2388.1869
12.	2479.9995	2651.6749	2388.1495	2559.8426
13.	2660.0017	2831.6626	2559.8017	2731.4715
14.	2839.9986	3011.6432	2731.4486	2903.1125
15.	3019.9993	3191.6340	2903.0993	3074.7430
16.	3200.0012	3371.6290	3074.7512	3246.3835
17.	3379.9988	3551.6326	3246.3988	3418.0432
18.	3559.9972	3731.5970	3418.0472	3589.6530
19.	3739.9999	3911.6010	3589.6999	3761.3295
20.	3920.0012	4091.5772	3761.3512	3932.9417
21.	4100.0013	4271.5766	3933.0013	4104.5947
22.	4279.9995	4451.5819	4104.6495	4276.2454
23.	4460.0024	4631.6072	4276.3024	4447.8740
24.	4640.0018	4811.5961	4447.9518	4619.5475
25.	4819.9974	4991.6133	4619.5974	4791.2059
26.	5000.0011	5171.6140	4791.2511	4962.8673
27.	5179.9974	5351.6083	4962.8974	5134.5119
28.	5360.0023	5531.6340	5134.5523	5306.1747
29.	5539.9978	5711.6491	5306.1978	5477.8172
30.	5719.9976	5891.6616	5477.8476	5649.4904

	Example 3		Example 4	
	d_{off}	d_{on}	d_{off}	d_{on}
1.	237.795	432.222	237.795	432.222
2.	432.231	626.674	432.231	626.674
3.	626.667	821.123	626.667	821.123
4.	821.103	1015.584	821.103	1015.584
5.	1015.539	1210.039	1015.539	1210.039
6.	1209.975	1404.476	1209.975	1404.476
7.	1404.411	1598.890	1404.411	1598.890
8.	1598.847	1793.366	1598.847	1793.366
9.	1793.283	1987.758	1793.283	1987.758
10.	1987.719	2182.216	1987.719	2182.216
11.	2182.155	2376.644	2182.155	2376.644
12.	2376.591	2571.088	2376.591	2571.088
13.	2571.027	2765.472	2571.027	2765.472
14.	2765.463	2959.907	2765.463	2959.907
15.	2959.899	3154.327	2959.899	3154.327
16.	3154.335	3348.746	1185.825	1230.377
17.	3348.771	3543.194	1230.350	1274.882
18.	3543.207	3737.594	1274.875	1319.408
19.	3737.643	3932.029	1319.400	1363.948
20.	3932.079	4126.456	1363.925	1408.471
21.	4126.515	4320.874	1408.450	1452.977
22.	4320.951	4515.311	1452.975	1497.511
23.	4515.387	4709.780	1497.500	1542.053
24.	4709.823	4904.217	1542.025	1586.559
25.	4904.259	5098.638	1586.550	1631.080
26.	5098.695	5293.098	1631.075	1675.614
27.	5293.131	5487.543	1675.600	1720.153
28.	5487.567	5681.984	1720.125	1764.643
29.	5682.003	5876.463	1764.650	1809.199
30.	5876.439	6070.893	1809.175	1853.709

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